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# EARTHQUAKE ANALYSIS OF MULTISTOREYED BUILDINGS INCLUDING ROTATIONS OF FLOORS †

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## INTRODUCTION

A multistoreyed building in vibration is an extremely complicated system. It has infinite degrees of freedom. Finding precisely the motion of such a system except by actual measurements is beyound accomplishment. However, the system can he simplified by means of reasonable approximations to reduce it to a solvable mathematical model.

One of these simplifications is to assume the mass of the building to be concentrated at the floor levels. It is also assumed that the floors act as rigid diaphragms. Then neglecting the displacements of the floors out of their respective planes, the building can be considered to have three degrees of freedom per floor, viz. two translational at right angles to each other and one rotational[1]. The translational motion in two principal directions are usually considered independently of each other. In addition, if the centre of mass coincides with the centre of rigidity at each floor, the translational motion of the floor will occur without any rotation when the building is subjected to ground motion in one of the principal directions. Dynamic analysis of the building can then be carried out considering only one degree of freedom per floor.

If the building is symmetric in plan the centre of mass can be considered to coincide with the centre of rigidity at all floors. But in many practical cases, building is not symmetric in plan, and hence the centre of mass does not coincide with the centre of rigidity at various floor levels. In such cases dynamic analysis of the building must be carried out by taking into consideration rotations of the floors also i.e. by considering two degrees of freedom per floor. However, this analysis becomes somewhat complicated and hence. hitherto there has been a tendency to neglect the floor rotations when carrying out the dynamic analysis of a multistoreyed building subjected to earthquake motions.

In the present paper it is shown that the dynamic earthquake analysis of a multistoreyed building including floor rotations can be carried out by using a procedure very similar to the one used when rotations are not considered. It is also shown that the method of response spectrum can be very well applied in this case also, and that the response spectra developed for one degree of freedom systems can also be used in this case with a small modification in the definition of the participation factors.

Because of the similarity of the two procedures of calculating the response of structure with or without considering floor rotations, theoretical derivations are first given below when no rotations are considered. Subsequently, it is shown how this procedure could be modified for including the floor rotations also. A numerical example of a 2-storeyed shear building is considered for comparing the design forces obtained on various frames by the two methods viz. when floor rotations are considered and when they are neglected.

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## THEORETICAL DERIVATION NEGLECTING FLOOR ROTATIONS

Figure 1 shows the plan of a typical $r^{th}$ floor of a multistoreyed building having n floors and m frames per floor. Assuming that the centre of mass coincides with the centre of rigidity at each floor we can write the equation of motion of the building for free vibration in the X direction in matrix form as[2] :

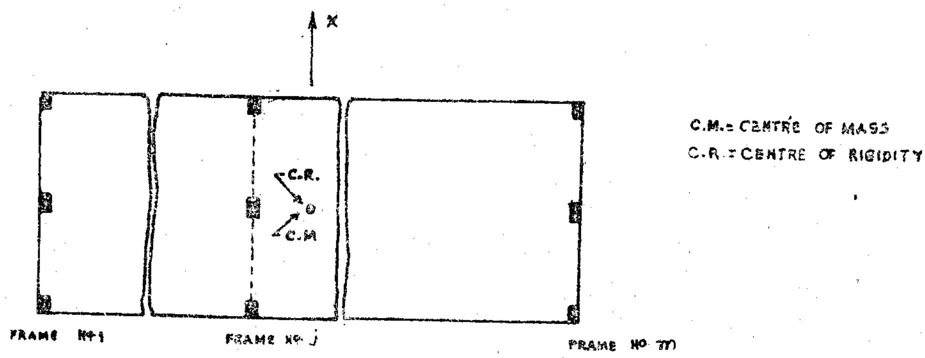


Figure 1. Typical Plan of $r^{th}$ Floor

$$[M]\{\ddot{q}\} + [K]\{q\} = 0 \tag{1}$$

Here $\{q\}$ is a column vector containing displacements $x_1, x_2$ — — — $x_n$ of the n floors. [M] is a mass matrix which is diagonal and [K] is the stiffness matrix.

If the building is now subjected to a ground motion $X_s$ ($\pm$) then equation (1) gets transformed to

$$[M]\{\ddot{u}\} + [K]\{u\} = -\ddot{X}_s [M]\{i\} \tag{2}$$

where $\{u\} = \{q\} - X_s \{i\}$ (3)

and contains the relative displacements of floors with respect to ground and $\{i\}$ is a unit vector.

Equation (2) is a set of coupled second order differential equations. To uncouple these equations we make a linear transformation

$$\{u\} = [\phi]\{\xi\} \tag{4}$$

where $[\phi]$ is a transformation maxtrix and $\{\xi\}$ are the new variables.

Substituting eqn. (4) in eqn. (2), we can write

$$\{\ddot{\xi}\} + [\Lambda]\{\xi\} = \ddot{X}_s [\phi]^{-1}\{i\} \tag{5}$$

where $[\Lambda] = [\phi]^{-1}[D][\phi]$ (6)

and $[D] = [M]^{-1}[K]$ (7)

Equation (5) will be uncoupled if we choose $[\phi]$ such that matrix $[\Lambda]$ becomes diagonal. It can be shown that this is achieved by constructing $[\phi]$ from the eigenvectors $\{X\}_i$ of matrix [D] as follows[2] :

$$[\phi] = [\{X\}_1 \; \{X\}_2 \; — — — \; \{X\}_n] \tag{8}$$

It is easy to prove that the elements of matrix $[\Lambda]$ are the squares of the natural frequencies of vibration of the building and vectors $\{X\}_1$ represent the mode shapes. Using the orthogonality relations of the vectors $\{X\}_1$, the unit vector $\{i\}$ on the right hand side of eqn. (2), can be expanded in terms of these eigenvectors as

$$\{i\} = [\{X\}_1 \{X\}_2 ---- \{X_n\}] (c)$$

or $$\{i\} = [\phi] \{c\} \tag{9}$$

where $\{c\}$ contains unknown constants $c_1$, $c_2$ --- $c_n$ which are also called as participation factors and are given by

$$c_r = \frac{\{X\}_r^T [M] \{i\}}{\{X\}_r^T [M] \{X_r\}} \tag{10}$$

Substituting eqn. (9) in equation (5) we get

$$\{\ddot{\xi}\} + [\Lambda] \{\xi\} = - \ddot{X}_s \{c\} \tag{11}$$

Equation (11) represents a set of uncoupled second order differential equations with right hand side containing the ground motion $X_s$ (t) multiplied by the participation factor $C_r$.

Hence the response of the building, say, in the $r^{th}$ mode for a given ground motion $X_t$ (t) will be same as that of a one degree freedom system, having the same period as that of $r^{th}$ mode, and hence the response spectra developed for one degree of freedom system can also be used for obtaining the response of the building in different individual modes, by multiplying by the participation factor. In the earthquake analysis of multistoreyed buildings the responses of the building in different modes are obtained separately and then a most probable response is obtained by taking the square root of the sum of squares of the response in the individual modes. We will show now that a similar technique can be used when the floor rotations are also taken into consideration.

## THEORETICAL DERIVATIONS WHEN FLOOR ROTATIONS ARE CONSIDERED

Figure 2 shows the plan of a typical $r^{th}$ floor of a multistoreyed building in which, now the centre of mass does not coincide with the centre of rigidity, and is having an eccentricity $e_r$. As before, the building has n floors and m resisting frames in the direction of ground motion. The distance of the frame j, from the centre of mass is denoted by $l_{r,j}$ ($j = 1, 2...m$) taken positive to the right of the centre of mass. It may be noted that if the centres of masses of all floors lie along same vertical line, a case which may be encountered in many practical buildings, $l_{r,j}$ will be same for all floors, $r = 1, ...n$. Each floor has now two degrees of freedom, viz. the displacements $x_r$ and rotations $\theta_r$. If the building has n storeys then it will have 2 n degrees of freedom.

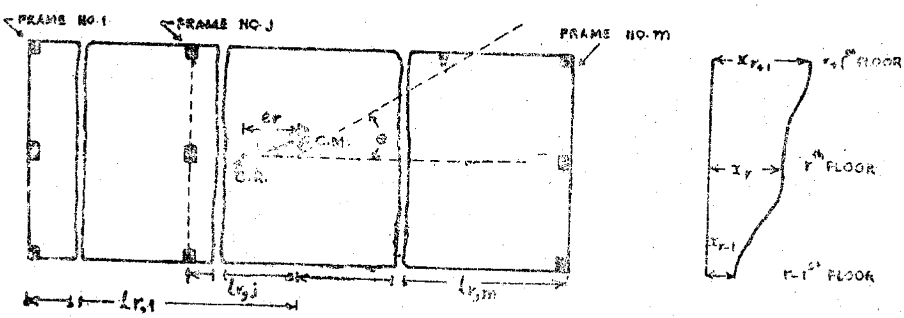


Fig. 2. Typical Plan and Elevation of $r^{th}$ Floor

The definitions of the stiffness coefficients are the same as before. When the structure is subjected to a ground motion in x direction all floors will translate as well as rotate and the displacement of the $j^{th}$ frame at $r^{th}$ floor will be given by equation

$$x_{r,j} = x_r + l_{r,j} \times \theta_r \tag{12}$$

where, $x_{r,j}$ = displacement at $r^{th}$ floor of $j^{th}$ frame.

$x_r$ = displacement of the centre of mass of $r^{th}$ floor

$\theta_r$ = rotation of the $r^{th}$ floor taken positive counter clockwise.

The equations of motion for the $r^{th}$ floor of a shear building can now be written as

$$M_r \ddot{x}_r = -\sum_{j=1}^{m} K_{r,j} (x_{r,j} - X_{r-1,j}) + \sum_{j=1}^{m} K_{r+1,j} (x_{r+1,j} - x_{r,j}) \tag{13}$$

and

$$I_r \ddot{\theta}_r + \sum_{j=1}^{m} K_{r,j} \times l_{r,j} (X_{r,j} - X_{r-1,j}) - \sum_{j=1}^{m} K_{r+1,j} \times l_{r+1,j} (x_{r+1,j} - x_{r,j}) = 0 \tag{13a}$$

Here, $I_r$ is the mass moment of interia of the $r^{th}$ floor about its centre of mass. We define

$$\sum_{j=1}^{m} K_{r,j} = K_r$$

where, $K_r$ = total stiffness of all frames at $r^{th}$ floor. Also, by definition of the centre of rigidity we can write

$$\sum_{j=1}^{m} K_{r,j} \times l_{r,j} = K_r \times e_r$$

where $e_r$ = eccentricity, i.e. distance between centre of mass and centre of rigidity at $r^{th}$ floor.

Similarly, we define

$$R_r = \sum_{j=1}^{m} K_{r,j} \times l^2_{r,j}$$

Substituting this and also eq. (12) in eqs. (13) and (13 a) we get

$$M_r \ddot{x}_r = K_r x_{r-1} - (K_r + K_{r+1}) x_r + K_{r+1} x_{r+1}$$

$$+ \theta_{r-1} \sum_{j=1}^{m} K_{r,j} l_{r,j} - \theta_r \left\{ K_r e_r + \sum_{j=1}^{m} K_{r+1,j} l_{r,j} \right\} + K_{r+1} e_{r+1} \theta_{r+1} \tag{14}$$

and

$$I_r \ddot{\theta}_r + [K_r e_r + K_{r+1} e_{r+1}] x_r - K_r e_r x_{r-1} - K_{r+1} e_{r+1} x_{r+1} - R_r \theta_r - R_{r+1} \theta_{r+1}$$

$$- \sum_{j=1}^{m} K_{r,j} l_{r,j} (l_{r-1,j} \theta_{r-1}) + \sum_{j=1}^{m} K_{r+1,j} l_{r+1,j} (l_{r,j} \theta_r) = 0 \tag{15}$$

These are the most general equations in the case of a 'shear' building. Now, if centres of masses of all floors lie on the same vertical line, then,

$$l_{1,j} = l_{2,j} \ldots \ldots l_{r,j} = l_{r+1,j} \ldots \ldots = l_{n,j}$$

Then, equations (14) and (15) get simplified to

$$M_r \ddot{x}_r = K_r x_{r-1} - (K_r + K_{r+1}) x_r + K_{r+1} x_{r+1}$$
$$+ K_r e_r \theta_{r-1} - (K_r e_r + K_{r+1} e_{r+1}) \theta_r + K_{r+1} e_{r+1} \theta_{r+1} \quad (16)$$

and

$$I_r \ddot{\theta}_r - K_r e_r x_{r-1} + (K_r e_r + K_{r+1} e_{r+1}) x_r - K_{r+1} e_{r+1} x_{r+1} - R_r \theta_{r-1}$$
$$+ (R_r + R_{r+1})\theta_r - R_{r+1} \theta_{r+1} = 0 \quad (17)$$

Similar equations can be written for all other floors and all these equations can then be combined into a single matrix equation similar to eq. (1c).

$$[M]\{\ddot{q}\} + [K]\{q\} = 0 \quad (18)$$

where [M], [K] and {q} are as shown in Table 1. It may be noted that the top left partition matrix of [K] in Table 1 is same as the stiffness matrix [K] of eq. (1) for the shear building without floor rotations. Matrices [M] and [K] can be readily obtained when the dimensions etc. of a shear building are known. For a flexible building in which joint rotations are to be considered eqs. (13) and (13 a) will contain on their right hand side $x_r$, $\theta_r$ and $K_{r,j}$ of all floors ($r = 1 \ldots \ldots n$). However, the final form of the equations will be same as eq. (18). The stiffness matrix [K] in this case will be different from that given in Table 1. The details of this are at present being worked out by the authors. If the building is now subjected to a ground motion $X_s$ (t) in the x direction, the equation of motion (18) becomes

$$[M](\ddot{q}) + [K]\{q\} - X_s\{i_o\} = 0 \quad (19)$$

where $\{i_o\}$ is now a vector which contains unity for first n terms corresponding to displacements $x_i$ and zeros for next n terms. If $u_1 \ldots \ldots u_n$ represent the displacements of the floors relative to ground i.e. if

$$u_r = x_r - X_s \ldots r = 1, \ldots n$$

and $$u_r = \theta_r \ldots r = n+1, \ldots \ldots 2n$$

Then, we can write

$$\{u\} = \{q\} - X_s\{i_o\}$$

substituting in eq. (19) we get

$$[M]\{\ddot{u}\} + [K]\{u\} = -\ddot{X}_s [M]\{i_o\} \quad (20)$$

Eq. (20) represents the equations of motion of a building subjected to ground motion when floor rotations are also considered. They have exactly the same form as eq. (2) except that all the matrices have now 2n rows and the vector {i} is now replaced by vector $\{i_o\}$.

Proceeding in exactly the same manner as before we can uncouple eqs. (20) and obtain an equation similar to eq. (11) :

$$\{\ddot{\xi}\} + [\Lambda]\{\xi\} = -\ddot{X}_s\{c\} \quad (21)$$

Here now the participation factors {c} are coefficients in the series expansion of the vector $\{i_o\}$ in terms of the new eigenvectors

$$c_r = \frac{\{X\}_r^T [M] \{i_o\}}{\{X\}_r^T [M] \{X\}_r}$$

Thus, for a known ground motion $X_s(t)$ we can once again obtain the response of the building in individual modes by using a known response spectra and the newly defined participation factors. The response as obtained by this method will give us the displacements of the centre of mass and rotations of each floor. For design purposes we require the forces exerted at floor levels of individual frames. These can be obtained as follows.

Knowing the response $X_r$ (of centre of mass) and $\theta_r$ for the $r^{th}$ floor in a particular mode we can obtain the displacements at $r^{th}$ floor for the $j^{th}$ frame by using eq. (12). The forces at floor level for individual frames in that particular mode can then be obtained by multiplying these displacements by the stiffness matrix of the frame. Such forces are obtained for all the modes, and the most probable forces can then be obtained by model superposition. In this way the most probable forces are obtained for all the frames.

## NUMERICAL EXAMPLE

To illustrate the method discussed above, earthquake analysis of two-storeyed shear building was carried out. The floor plan of the building is shown in Fig. 3. The building has three resisting frames for motion in the 'x' direction. These frames were assumed to have the same stiffness. However, since the frames are unsymmetrically placed, the centres of masses do not coincide with the centres of rigidity on both floors. The centres of masses, however, lie on the same vertical line. The earthquake response of the building was calculated by using the average acceleration spectrum given in Indian Standards Code for Earthquake Resistant Design of Structures[3]. The value of N was assumed to be unity.

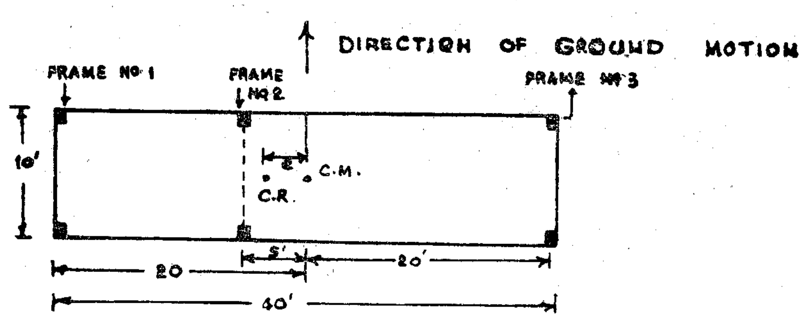


Figure 3. Plan

For comparison, the building was analysed both by neglecting the eccentricity of the centre of mass and by considering it i.e. by considering floor rotations. Knowing the loads on floors and the dimensions of the members, matrices [M] and [K] were obtained in each case. In first case when no floor rotations were considered the stiffness matrix was obtained from left portion matrix of Table 1 and in the second case it was formulated as per Table 1. The periods of vibrations, mode shapes, floor forces etc. were then calculated in each case. In the first case, the total floor force obtained was divided equally to the three frames, since these three frame have equal stiffness. While, in second case, the forces on each frame were obtained in each mode by the procedure discussed earlier, and then the most probable forces for each frame were obtained. Results are summarised in Tables 2 to 6. It can be seen that the periods of vibration increase, as excepted, when floor rotations are considered. The most probable total floor forces at the first and second floors in the first case are 4150 and 10,000 lbs respectively. While in the second case, when floor rotations are considered these values are 3950 and 9850 respectively. It can be seen that the total floor forces almost remain the same, whether floor rotations are considered or not. The small differences are due to the fact that when floor rotations are considered, the periods of vibrations change and this gives rise to slightly different value of responses from the response spectrum.

TABLE 1

$$
\left[\begin{array}{ccccc|ccccc}
M_1 & 0 & 0 & - & 0 & & & & & \\
0 & M_2 & 0 & - & 0 & & & & & \\
0 & 0 & M_3 & - & 0 & & & & & \\
- & - & - & - & - & & & & & \\
0 & 0 & 0 & - & M_n & & & & & \\
\hline
 & & & & & I_1 & 0 & 0 & - & 0 \\
 & & & & & 0 & I_z & 0 & - & 0 \\
 & & & & & 0 & 0 & I_3 & - & 0 \\
 & & & & & - & - & - & - & - \\
 & & & & & 0 & 0 & 0 & - & I_n
\end{array}\right]
\left\{\begin{array}{c}
\ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ - \\ \ddot{x}_n \\ \hline \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ - \\ \ddot{\theta}_n
\end{array}\right\}
+
\left[\begin{array}{ccccc|ccccc}
K_1+K_2 & -K_2 & 0 & - & 0 & K_1e_1+K_2e_2 & -K_2e_2 & 0 & - & 0 \\
-K_2 & K_2+K_3 & -K_3 & - & 0 & -K_2e_2 & K_2e_2+K_3e_3 & -K_3e_3 & - & 0 \\
0 & -K_3 & K_3+K_4 & - & 0 & 0 & -K_3e_3 & K_3e_3+K_4e_4 & - & 0 \\
- & - & - & - & - & - & - & - & - & - \\
0 & 0 & 0 & 0 & K_n & 0 & 0 & 0 & - & K_ne_n \\
\hline
 & & & & & R_1+R_2 & -R_2 & & - & 0 \\
 & & & & & -R_2 & R_2+R_2 & -R_3 & - & 0 \\
 & & \text{SYMMETRIC} & & & 0 & -R_3 & R_3+R_4 & - & 0 \\
 & & & & & - & - & - & - & - \\
 & & & & & 0 & 0 & 0 & - & R_n
\end{array}\right]
\left\{\begin{array}{c}
x_1 \\ x_2 \\ x_3 \\ - \\ x_n \\ \hline \theta_1 \\ \theta_2 \\ \theta_3 \\ - \\ \theta_n
\end{array}\right\}
= 0
$$

$$[M]\{\ddot{q}\} + [K]\{q\} = 0$$

TABLE 2

Mode Shapes and Floor Forces without Floor Rotations

| Floor levels | First mode shape coefficient | Second mode shape coefficient | First mode forces in lbs | Second mode forces in lbs | Probable forces in lbs | Probable forces on frame |
|---|---|---|---|---|---|---|
| First floor | 0.4037 | 1.000 | 4001 | 4210 | 4150 | 1383 |
| Second floor | 1.0000 | –0.4037 | 9910 | –1700 | 10000 | 3333 |

First mode period = 0.1738 Seconds
Second mode period = 0.07197 Seconds

TABLE 3

Periods of Free Vibration
(when floor rotations are considered)

| Mode No. | Period in seconds |
|---|---|
| 1 | 0.1756 |
| 2 | 0.1241 |
| 3 | 0.0727 |
| 4 | 0.0514 |

TABLE 4

Normalised Mode Shape Coefficients
(When floor rotations are considered)

| Floor level | First mode | Second mode | Third mode | Fourth mode |
|---|---|---|---|---|
| 1 | 0.4037 | 0.4037 | 1.00 | 1.00 |
| 2 | 1.00 | 1.00 | –0.4037 | –0.4037 |
| 3 | 0.00493 | –-0.2329 | 0.0122 | –0.5769 |
| 4 | 0.0122 | –0.5769 | –0.00493 | 0.2329 |

TABLE 5
Storey Forces for Whole Structure : (When floor rotations are considered)

| | First mode | Second mode | Third mode | Fourth mode | Probable forces |
|---|---|---|---|---|---|
| 1st floor force | 3918 | 83 | 4122 | 87 | 3950 |
| 2nd floor force | 9705 | 206 | —1664 | —35 | 9850 |
| 1st floor moment | 6791 | —6791 | —7145 | —7145 | — |
| 2nd floor moment | 16821 | —16821 | —2885 | 2885 | — |

TABLE 6
Modal and Probable Floor Forces for Each Frame (When floor rotations are considered)

| | Frame No. 1 | | | | | Frame No. 2 | | | | | Frame No. 3 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1st mode | IInd mode | IIIrd mode | IVth mode | Probable | 1st mode | IInd mode | IIIrd mode | IVth mode | Probable | Ist mode | IInd mode | IIIrd mode | IVth mode | Probable |
| 1st floor force | 1007 | 177 | 1059 | 186 | 1050 | 1252 | 55 | 1317 | 58 | 1270 | 1659 | —149 | 1746 | —157 | 1720 |
| 2nd floor force | 2494 | 439 | —428 | —75 | 2560 | 3100 | 136 | —532 | —23 | 3150 | 4110 | —369 | —705 | 63 | 4170 |

When no floor rotations are considered the forces on the three frames are the same viz. 1383, and 3333 lbs at the first and second floor respectively. However, when floor rotations are considered, Table 6 shows that the forces on the frames vary. At the first floor level they are 1050, 1270 and 1720 lbs and at second floor level they are 2560, 3150 and 4170 lbs for frame numbers 1, 2 and 3 respectively. Thus, when floor rotations are considered, as expected, frame 3 carries much greater forces than the other two frames and also than that what were obtained by neglecting floor rotations. Hence, we can summarise that in the dynamic earthquake analysis of an unsymmetric building, floor rotations should be taken into consideration. Although this will give rise to a total design floor force of almost the same magnitude as that obtained when floor rotations are neglected its distribution to the individual frames is quite different. In such cases the frames on the side of the centre of mass should be designed for greater loads than would be obtained by neglecting floor rotations.

## CONCLUSIONS

A method of finding the earthquake response of multistoreyed buildings, considering rotations of the floors has been developed. An example was considered to compare the design forces obtained by this method with those obtained when floor rotations are neglected as is done often in practice. The following conclusions can be drawn :

1. The equations of motions for vibrations of multistoreyed buildings when floor rotations are considered (i.e. when each floor has two degrees of freedom) are similar in nature to those when rotations are neglected. The dynamic analysis in this case can be carried out with equal simplicity as when floor rotations are neglected. Only the definitions of mass matrix [M] and stiffness matrix [K] are to be modified.

2. The response of multistoreyed buildings to ground motions can be found by a procedure similar to the one used when rotations are neglected. Only the definition of participations factors has to be modified.

3. The total floor forces through the centres of masses when floor rotations are considered are practically equal to the floor forces when such floor rotations are not considered.

4. In the case when floor rotations are considered the distribution of total floor forces amongst the individual frames is unequal. The frames which are farthest from the centre of mass will carry maximum or minimum floor forces depending on the location of centre of rigidity. This distribution is quite different from that obtained when floor rotations are neglected. Frames on the side of the centre of mass should be designed for greater forces when floor rotations are considered.