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DYNAMIC BEHAVIOUR OF CANTILEVER BEAMS

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Synopsis

A number of structures, like chimneys and stacks act as cantilever beams. This paper deals with the dynamic behaviour of such beams taking into account bending, shear and rotatory inertia deformations. Equations of motion have been given. Theoretical and numerical solutions are discussed. Natural frequencies of vibration have been determined for linearly tapering beams. First four modes of vibration have been determined. Slenderness ratio and taper have been varied. The variation of frequencies as a function of the various parameters have been presented in the form of graphs.

1—Introduction

Cantilevered structures like chimneys are generally tapered and have different slenderness ratios depending upon their height and material of construction. The behaviour of tapered cantilever beams subjected to either pure bending[1]† or pure shear[2] deformations have already been worked out. However, a beam under vibration has bending, shear and rotatory inertia deformations acting simultaneously[3,4,5].

In this study, natural frequencies of vibration, corresponding to first four modes of vibration, have been determined for tapered cantilever beams. The slenderness ratio and taper have been varied. Also, solid and hollow sections have been considered. The results have been expressed in the form of graphs. A comparison has been made with pure shear and pure bending cases. The natural frequency of vibration of beams has a value closer to pure bending case corresponding to lower modes of vibration and larger slenderness ratio and tends to those of pure shear case for higher modes of vibration and smaller slenderness ratio.

2—Basic Equations of the Problem

The following assumptions are made in solving the problem. The material of which the beam is made is homogeneous, isotropic and behaves elastically. The cross sections remain plane in bending. The cross section in shear are free to warp but the resulting axial inertial forces are neglected. Only small oscillations are to be considered. A beam model satisfying the above assumption is generally known as Timoshenko Beam.

Considering the rotational and translational equilibrium of elastic and inertial forces[4], we have

$$V = -\sigma AG \frac{\partial y_s}{\partial x} \qquad 2.1‡$$

$$M = EI \frac{\partial^2 y_b}{\partial x^2} \qquad 2.2$$

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† Refers to serial number of references listed at the end.

‡The letter symbols adopted for use in this paper are listed alphabatically in appendix.

$$\frac{\partial M}{\partial x} = V+\rho I \frac{\partial^2 \theta_b}{\partial t^2} = V + \rho I \frac{\partial^3 y_b}{\partial x \delta t^2} \qquad 2.3$$

$$\frac{\partial V}{\partial x} = -m \frac{\partial^2 y}{\partial t^2} + w(x,t) \qquad 2.4$$

$$y = y_b + y_s \qquad 2.5$$

where σ represents the ratio of average shear stress on a section to the product of the shear modulus and the angle of shear at the neutral axis.

Combining 2.1, 2.2 and 2.3, we get

$$\frac{\partial}{\partial x}\left(EI \frac{\partial^2 y_b}{\partial x^2}\right) = -\sigma AG \frac{\partial y_s}{\partial x} + \rho I \frac{\partial^3 y_b}{\partial x \partial t^2} \qquad 2.6$$

Combining 2.1 and 2.4, we get

$$-\frac{\partial}{\partial x}\left(\sigma AG \frac{\partial y_s}{\partial x}\right) = -\rho A \frac{\partial^2 y}{\partial t^2} + w(x,t) \qquad 2.7$$

Equations 2.6 and 2.7 form a pair of coupled equations in y_b and y_s and these are the basic equations of the problem.

The above equations can also be derived from considerations of energy utilising Hamilton's principle. This will incidentally generate the boundary conditions[6].

3—Free Vibration Problem

For the sake of simplicity, we would consider the case of a uniform beam in obtaining a solution to the problem. However, all the general conclusions arrived at are also applicable to the case of non-uniform beams.

For the free vibrations of a uniform beam, considering a harmonic solution in time, equations 2.6 and 2.7 reduces to

$$EI \frac{d^3 y_b(x)}{dx^3} + \sigma AG \frac{dy_s(x)}{dx} + \rho Ip^2 \frac{dy_b(x)}{dx} = 0 \qquad 3.1$$

$$\sigma AG \frac{d^2 y_s(x)}{dx^2} + \rho Ap^2 (y_s(x) + y_b(x)) = 0 \qquad 3.2$$

Let $y_b(x) = B_b e^{\beta x}$ 3.3

and $y_s(x) = B_s e^{\beta x}$ 3.4

Substituting 3.3 and 3.4 in equations 2.1 and 2.3 we get

$$EI \beta^3 B_b + \sigma AG\beta B_s + \rho Ip^2 \beta B_b = 0 \qquad 3.5$$

and

$$\sigma AG\beta^2 B_s + \rho Ap^2 (B_b + B_s) = 0 \qquad 3.6$$

Rearranging 3.5 and 3.6 we get

$$(\beta^2 + (\rho/E)\, p^2)\, B_b + \left(\frac{\sigma G}{E} \cdot \frac{A}{I}\right) B_s = 0 \qquad 3.7$$

$$((\rho/E)\, p^2)\, B_b + \left((\rho/E)\, p^2 + \beta^2 \frac{\sigma G}{E}\right) B_s = 0 \qquad 3.8$$

In order that a solution of the form 3.3 and 3.4 may exist the determinant of the coefficients of equations 3.7 and 3.8 must vanish.

This yields a frequency equation of the form

$$p^4 - p^2\left(\frac{\sigma G}{\rho} \cdot \frac{A}{I} - \left(E/\rho + \frac{\sigma G}{\rho}\right) \beta^2\right) + \frac{E}{\rho} \cdot \frac{\sigma G}{\rho} \cdot \beta^4 = 0 \qquad 3.9$$

that is

$$p^2 = \tfrac{1}{2}\left(\frac{\sigma G}{\rho} \cdot \frac{A}{I} - \left(\frac{E}{\rho} + \frac{\sigma G}{\rho}\right)\beta^2\right)$$

$$\pm \sqrt{\tfrac{1}{4}\left(\frac{\sigma G}{\rho}\frac{A}{I} - \left(E/\rho + \frac{\sigma G}{\rho}\right)\beta^2\right)^2 - E/\rho \cdot \frac{\sigma G}{\rho} \cdot \beta^4} \qquad 3.10$$

Another equation in p and β can be had when the boundary conditions are applied to the problem and thus we can solve for p and β. In general, two values of β^2 correspond to every value of p^2. As equations 3.1 and 3.2 are fourth order in time, there will be two sets of real frequencies, one corresponding to the positive sign and the other to the negative sign of equation 3.10.

In general, the free vibration problem has a solution of the form

$$y_b\ (x, t) = \sum_{n1=1}^{\infty} \phi_{n_1}(x)\, B_{n1} \sin(p_{n1}t + \theta_{n1})$$

$$+ \sum_{n2=1}^{\infty} \phi_{n_2}(x)\, B_{n_2} \sin(p_{n_2} t + \theta_{n_2}) \qquad 3.11$$

$$y_s\ (x, t) = \sum_{n1=1}^{\infty} \psi_{n1}(x)\, B_{n_1} \sin(p_{n1} t + \theta_{n1})$$

$$+ \sum_{n2=1}^{\infty} \psi_{n_2}(x)\, B_{n_2} \sin(p_{n_2} t + \theta_{n_2}) \qquad 3.12$$

where

$$\phi_{n1}(x) + \psi_{n_1}(x) = Y_{n1}(x) \qquad 3.13$$

is the mode shape corresponding to first set of frequencies and

$$\phi_{n_2}(x) + \psi_{n_2}(x) = Y_{n_2}(x) \qquad 3.14$$

is the mode shape corresponding to second set of frequencies.

4—Forced Vibration Problem

Once again, for the sake of simplicity, we will consider the case of a uniform beam.

Equations 2.6 and 2.7 represent the equations of motion for the forced vibration problem and in the case of the uniform beam they reduce to

$$EI \frac{\partial^3 y_b}{\partial x^3} + \sigma AG \frac{\partial y_s}{\partial x} - \rho I \frac{\partial^3 y_b}{\partial x \partial t^2} = 0 \qquad 4.1$$

$$\sigma AG \frac{\partial^2 y_s}{\partial x^2} - \rho A \frac{\partial^2 (y_b + y_s)}{\partial t^2} = - w(x, t) \qquad 4.2$$

We assume a solution of the form

$$y_b(x, t) = \sum_{n1=1}^{\infty} \xi_{n1} \phi_{n1}(x) + \sum_{n2=1}^{\infty} \xi_{n2} \phi_{n2}(x) \qquad 4.3$$

$$y_s(x, t) = \sum_{n1=1}^{\infty} \xi_{n1} \psi_{n1}(x) + \sum_{n2=1}^{\infty} \xi_{n2} \psi_{n2}(x) \qquad 4.4$$

where ξ_{n1} and ξ_{n2} are harmonic functions of time.

Substituting 4.3 and 4.4 in 4.1 we get

$$\sum_{n1=1}^{\infty} \xi_{n1} \left(EI \frac{d^3 \phi_{n1}}{dx^3} \right) + \sum_{n1=1}^{\infty} \xi_{n1} \left(\sigma AG \frac{d\psi_{n1}}{dx} \right) - \sum_{n1=1}^{\infty} \left(\rho I \frac{d\phi_{n1}}{dx} \right) \ddot{\xi}_{n1}$$

$$+ \sum_{n2=1}^{\infty} \xi_{n2} \left(EI \frac{d^3 \phi_{n2}}{dx^3} \right) + \sum_{n2=1}^{\infty} \xi_{n2} \left(\sigma AG \frac{d\psi_{n2}}{dx} \right) - \sum_{n2=1}^{\infty} \left(\rho I \frac{d\phi_{n2}}{dx} \right) \ddot{\xi}_{n2} = 0 \qquad 4.5$$

Rearranging 4.5,

$$\sum_{n1=1}^{\infty} \xi_{n1} \left(EI \frac{d^3 \phi_{n1}}{dx^3} + \sigma AG \frac{d\psi_{n1}}{dx} \right) - \sum_{n1=1}^{\infty} \ddot{\xi}_{n1} \left(\rho I \frac{d\phi_{n1}}{dx} \right)$$

$$+ \sum_{n2=1}^{\infty} \xi_{n2} \left(EI \frac{d^3 \phi_{n2}}{dx^3} + \sigma AG \frac{d\psi_{n2}}{dx} \right) - \sum_{n2=1}^{\infty} \ddot{\xi}_{n2} \left(\rho I \frac{d\phi_{n2}}{dx} \right) = 0 \qquad 4.6$$

Since ϕ_{n1}, ϕ_{n2}, ψ_{n1} and ψ_{n2} are solutions to the free vibration problem, substituting these in equations 3.1 we get

$$EI \frac{d^3 \phi_{n1}}{dx^3} + \sigma AG \frac{d\psi_{n1}}{dx} = - \rho I p^2_{n1} \frac{d\phi_{n1}}{dx} \qquad 4.7$$

and

$$EI \frac{d^3 \phi_{n2}}{dx^3} + \sigma AG \frac{d\psi_{n2}}{dx} = - \rho I p^2_{n2} \frac{d\phi_{n2}}{dx} \quad 4.8$$

Making use of relation 4.7 and 4.8, equation 4.6 reduces to

$$\sum_{n1=1}^{\infty} (\ddot{\xi}_{n1} + p^2_{n1} \xi_{n1}) \rho I \frac{d\phi_{n1}}{dx} + \sum_{n2=1}^{\infty} (\ddot{\xi}_{n2} + p^2_{n2} \xi_{n2}) \rho I \frac{d\phi_{n2}}{dx} = 0 \quad 4.9$$

Similarity substituting 4.3 and 4.4 in 4.2 and making use of equation 3.2, we get

$$\sum_{n1=1}^{\infty} (\ddot{\xi}_{n1} + p^2_{n1} \xi_{n1}) \rho A (\phi_{n1} + \psi_{n1}) + \sum_{n2=1}^{\infty} (\ddot{\xi}_{n2} + p^2_{n2} \xi_{n2}) \rho A (\phi_{n2} + \psi_{n2}) = w(x, t) \quad 4.10$$

For proceeding with the solution of equations 4.9 and 4.10, we require an orthogonality relationship. The form of this relation is not the usual classical orthogonality condition. This condition is of the following form[7]

$$\int_0^L (\rho A (y_b + y_s)_m (y_b + y_s)_n + \rho I \left(\frac{dy_b}{dx}\right)_m \left(\frac{dy_b}{dx}\right)_n) dx = 0 \quad 4.11$$

The subscripts m and n refer to two separate solutions.

Let,

$$\sum_{n1=1}^{\infty} \rho I a_{n1} \frac{d\phi_{n1}}{dx} + \sum_{n2=1}^{\infty} \rho I a_{n2} \frac{d\phi_{n2}}{dx} = 0 \quad 4.12$$

and

$$\sum_{n1=1}^{\infty} \rho A Y_{n1} a_{n1} + \sum_{n2=1}^{\infty} \rho A Y_{n2} a_{n2} = w(x, t) \quad 4.13$$

Using equations 4.9 and 4.10 and equating terms on either sides, we get

$$\ddot{\xi}_{n1} + p^2_{n1} \xi_{n1} = a_{n1} \quad 4.14$$

and

$$\ddot{\xi}_{n2} + p^2_{n2} \xi_{n2} = a_{n2} \quad 4.15$$

Now

$$\int_0^L (w(x, t)) Y_{n1} dx + \int_0^L (0) \frac{d\phi_{n1}}{dx} dx$$

$$= \sum_{m1=1}^{\infty} a_{m1} \int_0^L \rho A Y_{m1} Y_{n1} dx + \sum_{m1=1}^{\infty} a_{m1} \int_0^L \rho I \frac{d\phi_{m1}}{dx} \frac{d\phi_{n1}}{dx}$$

$$+ \sum_{m2=1}^{\infty} a_{m2} \int_0^L \rho A Y_{m2} Y_{n1} dx + \sum_{m2=1}^{\infty} a_{m2} \int_0^L \rho I \frac{d\phi_{m2}}{dx} \frac{d\phi_{n1}}{dx} \quad 4.16$$

Using orthogonality relationship 4.11, we get

$$a_{n1} = \frac{\int_0^L w\,(x,\,t)\;Y_{n1}\;dx}{\int_0^L \left(\rho A\;Y_{n1}^2 + \rho\,I\left(\frac{d\phi_{n1}}{dx}\right)^2\right) dx} \tag{4.17}$$

Similarly

$$a_{n2} = \frac{\int_0^L w\,(x,\,t)\;Y_{n2}\;dx}{\int_0^L \left(\rho\,A\;Y_{n2}^2 + \rho\,I\left(\frac{d\phi_{n2}}{dx}\right)^2\right) dx} \tag{4.18}$$

Now solving 4.14 we get

$$\xi_{n1} = \frac{1}{p_{n1}} \int_0^t \frac{\int_0^L w\,(x,\,t)\;Y_{n1}\;dx}{\int_0^L \left(\rho\,A\;Y_{n1}^2 + \rho\,I\left(\frac{d\phi_{n1}}{dx}\right)^2\right) dx} \sin\;p_{n1}\,(t-\tau)\;d\tau \tag{4.19}$$

and from 4.15

$$\xi_{n2} = \frac{1}{p_{n2}} \int_0^t \frac{\int_0^L w\,(x,\,t)\;Y_{n2}\;dx}{\int_0^L \left(\rho\,A\;Y_{n2}^2 + \rho\,I\left(\frac{d\phi_{n2}}{dx}\right)^2\right) dx} \sin\;p_{n2}\,(t-\tau)\;d\tau \tag{4.20}$$

The final solution making use of 4.3, 4.4, 4.19 and 4.20 is

$$y\,(x,t) = y_b\,(x,\,t) + y_s\,(x,\,t)$$

$$= \sum_{n1=1}^{\infty} \frac{Y_{n1}(x)}{p_{n1}} \frac{\int_0^t \int_0^L (w\,(x,\,t)\,.\,Y_{n1}\,.\,dx)\,\sin\,p_{n1}\,(t-\tau)\;d\tau}{\int_0^L \left(\rho A\;Y_{n2}^2 + \rho I\left(\frac{d\phi_{n2}}{dx}\right)^2\right) dx}$$

$$+ \sum_{n2=1}^{\infty} \frac{Y_{n2}\,(x)}{p_{n2}} \frac{\int_0^t \int_0^L (w\,(x,\,t)\,.\,Y_{n2}\,.\,dx)\,\sin\,p_{n2}\,(t-\tau)\;dx}{\int_0^L \left(\rho\,A\;Y_{n2}^2 + \rho\,I\left(\frac{d\phi_{n2}}{dx}\right)^2\right) dx} \tag{4.21}$$

5—Earthquake Excitation

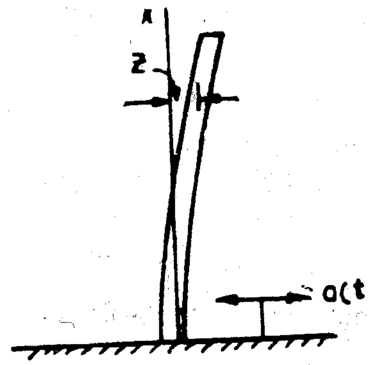

Fig. 1

Consider the case of a cantilever beam subjected to a transverse base motion described by the base acceleration a (t) as shown in Fig. 1.

If $\quad Z(x,t) = Z_b(x,t) + Z_s(x,t)$

is the relative motion between the beam element and the base, then the equation of motion becomes

$$\frac{\partial}{\partial x}\left(EI\,\frac{\partial^2 Z_b}{\partial x^2}\right) = -\sigma AG\,\frac{\partial Z_s}{\partial x} + \rho I\,\frac{\partial^3 Z_b}{\partial x\,\partial t^2} \qquad 5.1$$

and

$$-\frac{\partial}{\partial x}\left(\sigma AG\,\frac{\partial Z_s}{\partial x}\right) = -\rho A\left(\frac{\partial^2 Z}{\partial t^2} + a(t)\right) \qquad 5.2$$

Comparing 5.1 and 5.2 with 2.6 and 2.7, we see that y corresponds Z and w (x, t) to $-\rho A\,a(t)$ and therefore corresponding to 4.21 we have a solution for Z (x, t) of the form

$$Z(x,t) = \sum_{n1=1}^{\infty} \frac{Y_{n1}(x)\cdot\int_0^L \rho A\,Y_{n1}(x)\,dx}{\int_0^L\left(\rho A\,(Y_{n1}(x))^2 + \rho I\left(\frac{d\phi_{n1}}{dx}\right)^2\right)dx}\;\frac{1}{p_{n1}}\int_0^L a(t)\sin p_{n1}(t-\tau)\,d\tau$$

$$+\sum_{n2=1}^{\infty} \frac{Y_{n2}(x)\cdot\int_0^L \rho A\,Y_{n2}(x)\,dx}{\int_0^L\left(\rho A\,(Y_{n2}(x))^2 + \rho I\left(\frac{d\phi_{n2}}{dx}\right)^2\right)dx}\;\frac{1}{p_{n2}}\int_0^L a(t)\sin p_{n2}(t-\tau)\,d\tau \qquad 5.3$$

Let, the mode factor be denoted by **C**, where

$$C = \frac{1}{p}\,\frac{\int_0^L \rho A\,.\,Y(x)\,.\,dx}{\int_0^L\left(\rho A\,(Y(x))^2 + \rho I\left(\frac{d\phi}{dx}\right)^2\right)dx} \qquad 5.4$$

and the psuedo spectral response velocity by S_v, where

$$S_v = \left(\int_0^t a(t)\sin p\,(t-\tau)\,d\tau\right)_{maximum} \qquad 5.5$$

The maximum value of Z is then given by

$$Z_m(x) = \sum_{n1=1}^{\infty} Y_{n1}(x)\,.\,C_{n1}\,.\,S_{v_{n1}} + \sum_{n2=1}^{\infty} Y_{n2}(x)\,.\,C_{n2}\,S_{v_{n2}} \qquad 5.6$$

The relative displacement corresponding to any particular mode, say the rth mode, is given by

$$Z_r(x) = Y_r(x) \,.\, C_r \,.\, S_{v_r} \qquad 5.7$$

Similarly, the maximum bending moment at any particular point in the beam corresponding to r^{th} mode is given by

$$M_r(x) = EI(x) \frac{d^2\phi_r(x)}{dx^2} \,.\, C_r \,.\, S_{v_r} \qquad 5.8$$

and the maximum shear force is given by

$$V_r(x) = -\sigma AG(x) \frac{d\psi_r(x)}{dx} C_r \,.\, S_{v_r} \qquad 5.9$$

6—Theoretical Solution

The present study deals with cantilever beams. As will be indicated later, the solution gets very much involved even in the case of a uniform beam. It turns out, that only in the case of simply supported beam, the solution is straight forward as the shape functions are harmonic[3].

Consider a uniform beam and assume a solution to equation 2.1 and 2.2 of the form

$$Y_b = B_{1b}e^{\beta x} + B_{2b}e^{-\beta x} + B_{3b}e^{\beta' x} + B_{4b}e^{-\beta' x} \qquad 6.1$$

$$Y_s = B_{1s}e^{\beta x} + B_{2s}e^{-\beta x} + B_{3s}e^{\beta' x} + B_{4s}e^{-\beta' x} \qquad 6.2$$

There are apparently eight arbitrary constants but it can be shown that tnese are related in such a way that only four arbitrary constants exist and they can be evaluated from the four boundary conditions in space corresponding to the fourth order differential equation in space.

Substituting 6.1 and 6.2 in 3.1 and 3.2 and equating terms, we get

$$B_{1s} = -\frac{1}{\sigma AG}(\rho I p^2 + \beta^2 EI)\, B_{1b} = -\frac{\rho A p^2}{(\rho A p^2 + \sigma AG\beta^2)} B_{1b} \qquad 6.3$$

$$B_{2s} = -\frac{1}{\sigma AG}(\rho I p^2 + \beta^2 EI)\, B_{2b} = -\frac{\rho A p^2}{(\rho A p^2 + \sigma AG\beta^2)} B_{2b} \qquad 6.4$$

$$B_{3s} = -\frac{1}{\sigma AG}(\rho I p^2 + \beta'^2 EI)\, B_{3b} = -\frac{\rho A p^2}{(\rho A p^2 + \sigma AG\beta'^2)} B_{3b} \qquad 6.5$$

$$B_{4s} = -\frac{1}{\sigma AG}(\rho I p^2 + \beta'^2 EI)\, B_{4b} = -\frac{\rho A p^2}{(\rho A p^2 + \sigma AG\beta'^2)} B_{4b} \qquad 6.6$$

From 6.3, 6.4, 6.5 and 6.6, we get the following frequency equations relating β and β' to p

$$p^4 - p^2\left(\frac{\sigma G}{\rho} \,.\, \frac{A}{I} - \left(E/\rho + \frac{\sigma G}{\rho}\right)\beta^2\right) + \frac{E}{\rho}\,\frac{\sigma G}{\rho}\,\beta^4 = 0 \qquad 6.7$$

$$p^4 - p^2\left(\frac{\sigma G}{\rho} \,.\, \frac{A}{I} - \left(E/\rho + \frac{\sigma G}{\rho}\right)\beta'^2\right) + \frac{E}{\rho}\,\frac{\sigma G}{\rho}\,\beta'^4 = 0 \qquad 6.8$$

Let us apply boundary conditions to the problem to solve for β.

Taking the origin at the built-in end, we have for all t, At the built-in end,

total deflection $y_s + y_b \mid_{x=o} = 0$ 6.9

bending slope $\left.\frac{\partial y_b}{\partial x}\right|_{x=o} = 0$ 6.10

At the free end, moment $\left.\frac{\partial^2 y_b}{\partial x^2}\right|_{x=L} = 0$ 6.11

shear $\left.\frac{\partial y_s}{\partial x}\right|_{x=L} = 0$ 6.12

Let $\delta = \frac{1}{\sigma AG}(\rho I p^2 + \beta^2 EI)$

and $\delta' = \frac{1}{\sigma AG}(\rho I p^2 + \beta'^2 EI)$

then, equations 6.3, 6.4, 6.5 and 6.6 may be written as

$$B_{1s} = -\delta B_{1b}$$
$$B_{2s} = -\delta B_{2b}$$
$$B_{3s} = -\delta' B_{3b}$$
$$B_{4s} = -\delta' B_{4b} \quad 6.13$$

now, making use of boundary conditions 6.9, 6.10, 6.11 and 6.12, we have

$$(B_{1b}+B_{1s}) + (B_{2b}+B_{2s}) + (B_{3b}+B_{3s}) + (B_{4b}+B_{4s}) = 0 \quad 6.14$$
$$\beta(B_{1b}-B_{2b}) + \beta'(B_{3b}-B_{4b}) = 0 \quad 6.15$$
$$\beta(B_{1s}\,e^{\beta L} - B_{2s}\,e^{-\beta L}) + \beta'(B_{3s}\,e^{\beta' L} - B_{4s}\,e^{-\beta' L}) = 0 \quad 6.16$$
$$\beta^2(B_{1b}\,e^{\beta L} + B_{2b}\,e^{-\beta L}) + \beta'^2(B_{3b}\,e^{\beta' L} + B_{4b}\,e^{-\beta' L}) = 0 \quad 6.17$$

Rearranging 6.14 to 6.17

$$B_{1b}(1-\delta) + B_{2b}(1-\delta) + B_{3b}(1-\delta') + B_{4b}(1-\delta') = 0 \quad 6.18$$
$$B_{1b}(\beta) + B_{2b}(-\beta) + B_{3b}(\beta') + B_{4b}(-\beta') = 0 \quad 6.19$$
$$B_{1b}(-\delta\beta\,e^{\beta L}) + B_{2b}(\delta\beta e^{-\beta L}) + B_{3b}(-\delta'\beta'\,e^{\beta' L}) + B_{4b}(\delta'\beta' e^{-\beta' L}) = 0 \quad 6.20$$
$$B_{1b}(\beta^2 e^{\beta L}) + B_{2b}(\beta^2 e^{-\beta L}) + B_{3b}(\beta'^2 e^{\beta' L}) + B_{4b}(\beta'^2 e^{-\beta' L}) = 0 \quad 6{\cdot}21$$

Hence, for 6.1 and 6.2 to be a solution of equations 3.1 and 3.2, the following frequency determinant must be zero

$$\begin{vmatrix} 1-\delta & 1-\delta & 1-\delta' & 1-\delta' \\ \beta & -\beta & \beta' & -\beta' \\ -\delta\beta\,e^{\beta L} & \delta\beta\,e^{-\beta L} & -\delta'\beta' e^{\beta' L} & \delta'\beta'\,e^{-\beta' L} \\ \beta^2 e^{\beta L} & \beta^2 e^{-\beta L} & \beta'^2 e^{\beta' L} & \beta'^2 e^{-\beta' L} \end{vmatrix} = 0 \quad 6.22$$

Solving 6.22, we get the frequency parameter β.

It will be seen that the determinant is rather involved and even in the case of a uniform beam, the theoretical solution is quite complex.

7—Numerical Solution

While discussing theoretical solution of equations 2.1 and 2.2, it was observed that even in the case of a uniform beam the solution is very much involved for boundary conditions other than simply supported.

Further if the non-uniformity of the beam is not defined by a regular function, theoretical solution is almost impossible to obtain even in the case of beam subjected to pure bending or pure shear deformations. Numerical techniques are, therefore, resorted to in solving such problems.

The method consists in replacing a continuous system with a discrete system by concentrating the mass distribution into an equivalent set of discrete point masses embedded in an ideal massless substance possessing the same elastic properties as the body simulated.

Error Analysis of the Numerical Approach

Errors are due to approximating an infinite degrees of freedom system to a finite degrees of freedom system and not due to numerical technique involved. One type of error involves the number of masses used. The other type involves the determination of equivalent masses and stiffnesses.

To make an error analysis one should know the exact values of quantities under investigation. The errors in period are evaluated here.

The present investigation is mainly concerned with cantilever beams. We have exact solution of periods in the case of cantilever beams subject to pure bending or pure shear deformations.

Error in finding equivalent masses

There are generally two procedures adopted in finding out equivalent masses. In one case, the mass of a segment in divided equally and concentrated at the ends (Figure 2a). In the other case the mass is concentrated at the centre of a segment (Figure 2b).

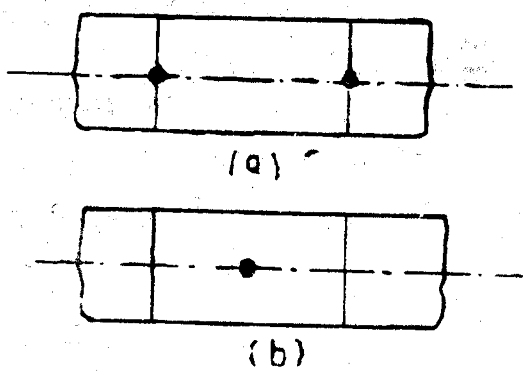

Fig. 2

It has been observed, by comparison with exact solution in the case of uniform bending beam, that errors are smaller if the mass is concentrated at the centre of segment.

Further, it has been shown that for a uniform shear beam, if the mass points are located at midpoints of equal segments, the error in frequency varies inversely as the square of number of segments, whereas, if the masses are placed at the ends the error varies as inverse first power of number of segments[8].

Error due to number of segments

In the case of a shear beam in addition to having an exact salution for the continuous system we also have an exact solution for the discrete system[9].

For the continuous system,

frequency parameter $\gamma_r = \frac{(2r-1)^2 \pi^2}{4}$

(γ is proportional to p^2)

For the discrete case with mass points concentrated at the middle of segments

$$\gamma_{rn} = 2n^2 \left(1 - \frac{\text{Cos}\,(2r-1)\,\pi}{2n} \right)$$

$$= \frac{(2r-1)^2 \pi^2}{4} - \frac{(2r-1)^4 \pi^4}{192\, n^2} + \text{small terms}$$

$$\text{The error } \epsilon_{rn} = \frac{\gamma_r - \gamma_{rn}}{\gamma_r} = \frac{(2r-1)^2 \pi^2}{48\, n^2} + \text{higher inverse powers of n}$$

This shows that error in γ for any given mode ultimately varies inversely as the square of number of segments and that proportional error for a given n increases rapidly.

In the case of a bending beam, we do not have an exact solution for the discrete system. Therefore, a numerical method, was adopted in calculating the frequency parameter. A uniform cantilever beam was divided into n equal segments where n ranged from five to hundred. The masses were assumed as concentrated at the centre of segments. In each case, the frequency parameter was obtained for the first four modes of vibration and the results compared with the exact solution for the infinite degrees of freedom system.

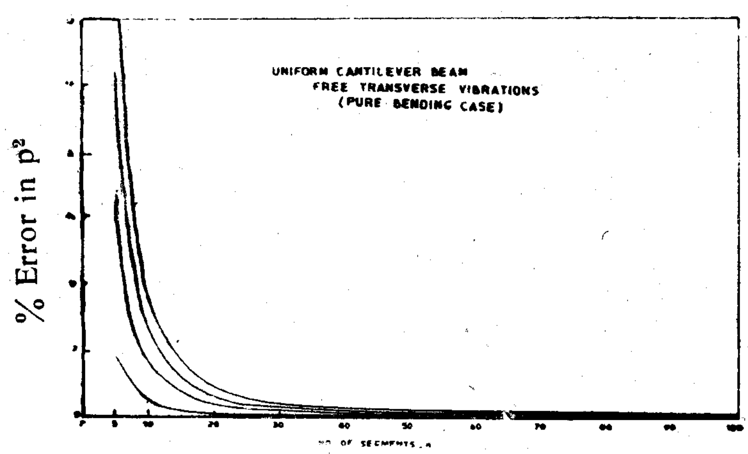

Fig. 3

Figure 3 shows a plot of ϵ_{rn} versus number of segments for the bending beam.

Modified Holzer Technique

Myklestad and Prohl were the first to adopt this numerical technique to the beam problem[10]. A small modification of their method is made here to take into account the characteristics of Timoshenko beam.

Consider equations 2.1 to 2.5 and assume that a harmonic solution in time with frequency p is applicable. Then, for the free vibration problem, we have

$$V = -\,\sigma A G \frac{dy_s}{dx} \tag{7.1}$$

$$M = EI \frac{d^2y_b}{dx^2} \quad 7.2$$

$$\frac{dM}{dx} = V - \rho I p^2 \theta_b \quad 7.3$$

$$\frac{dV}{dx} = mp^2y \quad 7.4$$

$$y = y_b + y_s \quad 7.5$$

Let the beam be divided into a a number of segments and one typical section of beam be as shown in figure 4.

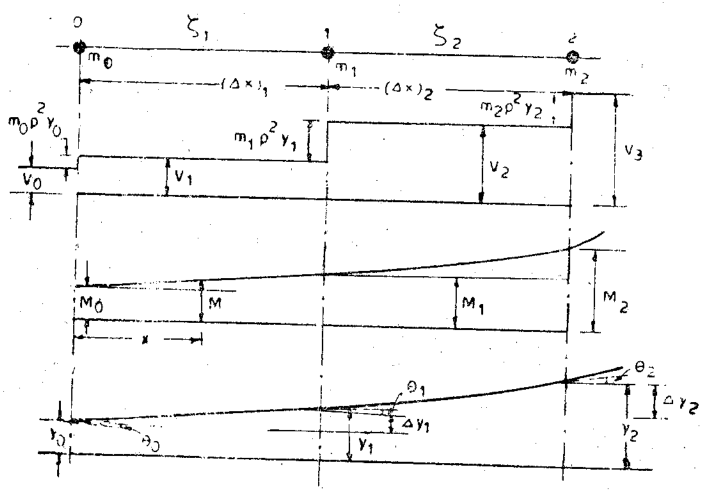

Fig. 4

A finite change of shear force occurs at each mass which is equal to inertia force of mass

$$\triangle V = mp^2y$$

Assume that the following quantities V_o, M_o, θ_{bo}, y_{bo} and y_{so} are known at the left section

Then

$$V_1 = V_o + m_o p^2 y_o \quad 7.6$$

$$M_1 = M_o + V_1 (\triangle x)_1 - (\rho I)_1 (\triangle x)_1 p^2 \theta_{bo} \quad 7.7$$

$$y_{s_1} = y_{s_o} - \left(\frac{V}{\sigma AG}\right)_1 (\triangle x)_1 \quad 7.8$$

Now, bending moment M at any distance x from thc left hand side section o is

$$M = M_o + \frac{M_1 - M_o}{(\triangle x)_1} x \quad 7.9$$

Slope
$$\theta_b = \frac{1}{(EI)_1} \int M dx + B \quad 7.10$$

$$= \frac{1}{(EI)_1} \left(M_o x + \frac{M_1 - M_o}{(\triangle x)_1} \frac{x^2}{2}\right) + \theta_{bo} \quad 7.11$$

and deflection $y_b = \int \theta dx + B'$ 7.12

$$= \frac{1}{(EI)_1} \left(M_o \frac{x^2}{2} + \frac{M_1 - M_o}{(\triangle x)_1} \frac{x^3}{6}\right) + \theta_{b_o} x + y_{b_o} \quad 7.13$$

Let
$$\left(\frac{\triangle x}{EI}\right) = \zeta \quad 7.14$$

then at $x = (\triangle x)_1$

$$\theta_{b_1} = \zeta_1 \left(\frac{M_o}{2} + \frac{M_1}{2}\right) + \theta_{b_o} \quad 7.15$$

and
$$y_{b_1} = \zeta_1 \left(\frac{M_o}{3} + \frac{M_1}{6}\right) (\triangle x)_1 + \theta_{bo} (\triangle x)_1 + y_o \quad 7.16$$

Now, we can generalise the result and write expressions for the nth section in terms of values at $(n-1)^{th}$ section.

$$V_n = V_{n-1} + m_{n-1}\, p^2\, y_{n-1} \qquad 7.17$$

$$M_n = M_{n-1} + V_n(\Delta x)_n - (\rho I)_n(\Delta x)_n\, p^2 \theta_{b_{n-1}} \qquad 7.18$$

$$\theta_{b_n} = \zeta_n \left(\frac{M_{n-1}}{2} + \frac{M_n}{2}\right) + \theta_{b_{n-1}} \qquad 7.19$$

$$y_{b_n} = \zeta_n \left(\frac{M_{n-1}}{3} + \frac{M_n}{6}\right) (\Delta x)_n + \theta_{b_{n-1}} (\Delta x)_n + y_{b_{n-1}} \qquad 7.20$$

$$y_{s_n} = y_{s_{n-1}} - \left(\frac{\Delta x}{\sigma G A}\right)_n V_n \qquad 7.21$$

$$y_n = y_{b_n} + y_{s_n} \qquad 7.22$$

Thus, we see that for any frequency p, once we know the values V, M, θ_b, y_s and y_b at a particular section we can find the corresponding values at all other sections.

Procedure for cantilever beam

At the built-in end, θ_b, y_b and y_s are equal to zero.

Let us choose an arbitrary value for p, say p′. First assume that a shear V′ (it can be 1.0 for convenience) and zero moment exists at built-in end and evaluate the shear and moment at the free end. Let it be B_1 and B_2. Next assume that a moment M′ (it can be 1.0 for convenience) and zero shear exists at built-in end and than evaluate the shear and moment at the free end. Let it be B_3 and B_4. Now if V_o and M_o are the actual values of shear and moment at the built-in end, then at the free end

$$V_e = B_1 \frac{V_o}{V'} + B_3 \frac{M_o}{M'} \qquad 7.23$$

and

$$M_e = B_2 \frac{V_o}{V'} + B_4 \frac{M_o}{M'} \qquad 7.24$$

Now if the value of p′ is such as to coincide with one of the natural frequencies of the system, then V_e and M_e would have been identically zero. That is, the determinant

$$\Delta = \begin{vmatrix} \dfrac{B_1}{V'} & \dfrac{B_3}{M'} \\ \dfrac{B_2}{V'} & \dfrac{B_4}{M'} \end{vmatrix} = 0 \qquad 7.25$$

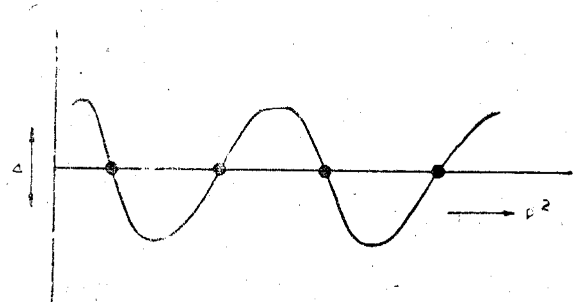

Fig. 5

In general, it would not be possible to guess the value of p correctly. However, we can arbitrarily assign various values to p′ and evaluate Δ. A plot of p^2 versus Δ would have a general appearance of fig. 5. The correct value of p^2 are those which correspond to intersection of the curve with the p^2 axis.

8. Description of Variables

For numerical analysis, in each case, the beam is divided into hundred segments. From the error analysis of shear and bending beams it is seen that error is negligibly small even upto fourth mode of vibration. As the Timoshenko beam lies between two extremes of shear and bending beam it is assumed that the errors would be negligible. The average mass of each segment is assumed to be concentrated at the middle of the segment. Average values over a segment are assumed to represent equivalent area and moment of inertia.

Taper Ratio α

Linearly tapering beams have been considered for analysis. The radius of gyration at any level x, where x is measured from free end, is given by, $r_x = k\left(a + \frac{b}{L}\right)$ where k, a and b are constants for a beam and its length is L. The ratio of radius of gyration at the free end to that at the fixed end has been designated as 'Taper Ratio', and is given by

$$\alpha = \frac{a}{a+b}$$

The value of α is varied from 0.2 to 1.0 and this covers a wide range of practical cases.

Parameter $\frac{E}{\sigma G}$

E and G are related by the well known expression

$$E = 2G(1+\nu)$$

where ν is the poisson's ratio of the material.

An excellent discussion on the value to be assumed for σ has been given by Mindlin[11]. The value of σ not only depends on the shape of the section but also on the mode of vibration. Further, if the beam is non-uniform, σ will also vary along the length of the beam For a solid section, a value of $\sigma = 5/6$ has been suggested. Taking a value of $\nu = 0.175$, for a solid section $\frac{E}{\sigma G}$ works out to 2.82.

Jacobsen has pointed out that σ may have a value of $\frac{1}{2.25}$ for a square box cross-section and $\frac{1}{2.58}$ for a square cellular section[12]. It appears that σ could vary from 5/6 for a solid section to 1/3 for a thin hollow section. Therefore, the parameter $E/\sigma G$ can have very different values and may not be even constant for a particular beam.

In this study, $E/\sigma G$ is considered constant for a particular beam and has a value of 2.82 corresponding to solid section and a value of 7.50 corresponding to thin hollow section.

Slenderness Ratio, L/r

The slenderness ratio is varied from 6.928 to 69.28. Corresponding to a solid section, this would mean a variation in length to depth ratio from 2 to 20. This covers a wide range of practical values.

The frequency of vibration has been expressed in the form

$$p = C_p \cdot \sqrt{E/\rho} \cdot \frac{1}{L} \cdot \frac{r_b}{L}$$

9—Discussion of Results and Conclusions

Figures 6 to 9 indicate the influence of slenderness ratio on the frequency corresponding to the first four modes of vibration of uniform beams. Frequency is inversely proportional to slenderness ratio for a bending beam and is independent of it for a shear beam. Frequency of a Timoshenko beam is less than either that of bending or of shear beam. Bending value and shear value form the two bounds of frequency for a beam. For large values of slenderness ratio, frequency tends to that of bending and for small values to that of shear. The influence of shear and rotatory deformations on frequency progressively increases with higher modes of vibration.

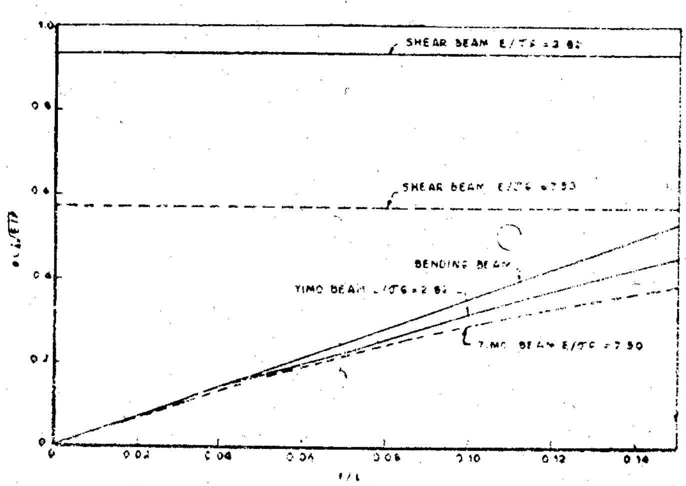

Fig. 6. Frequency vs Slenderness Ratio (1st Mode, Uniform Beam)

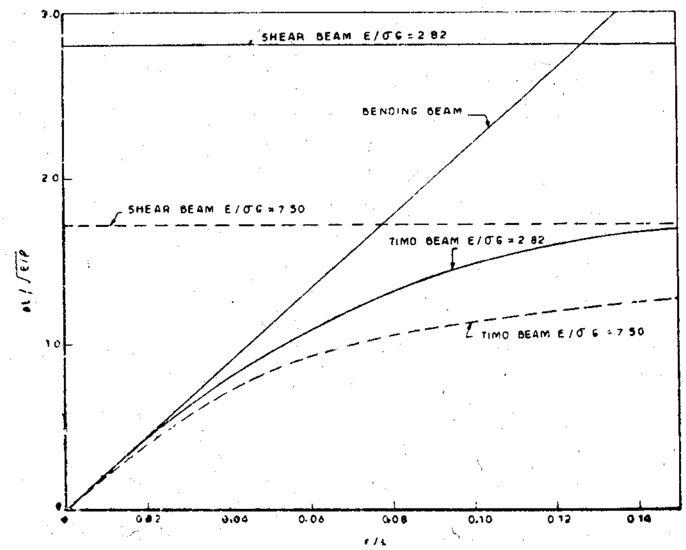

Fig. 7. Frequency vs Slenderness Ratio (2nd Mode, Uniform Beam)

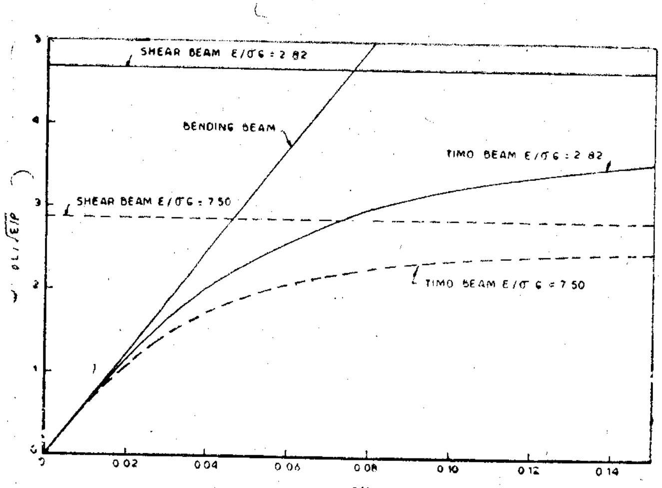

Fig. 8. Frequency vs Slenderness Ratio (3rd Mode, Uniform Beam)

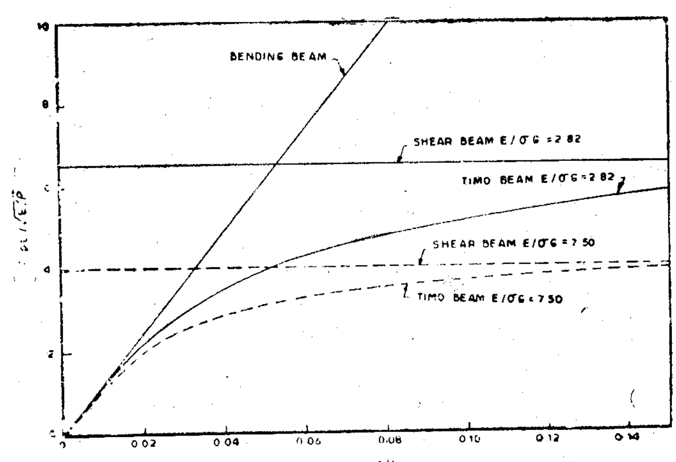

Fig. 9. Frequency vs Slenderness Ratio (4th Mode, Uniform Beam)

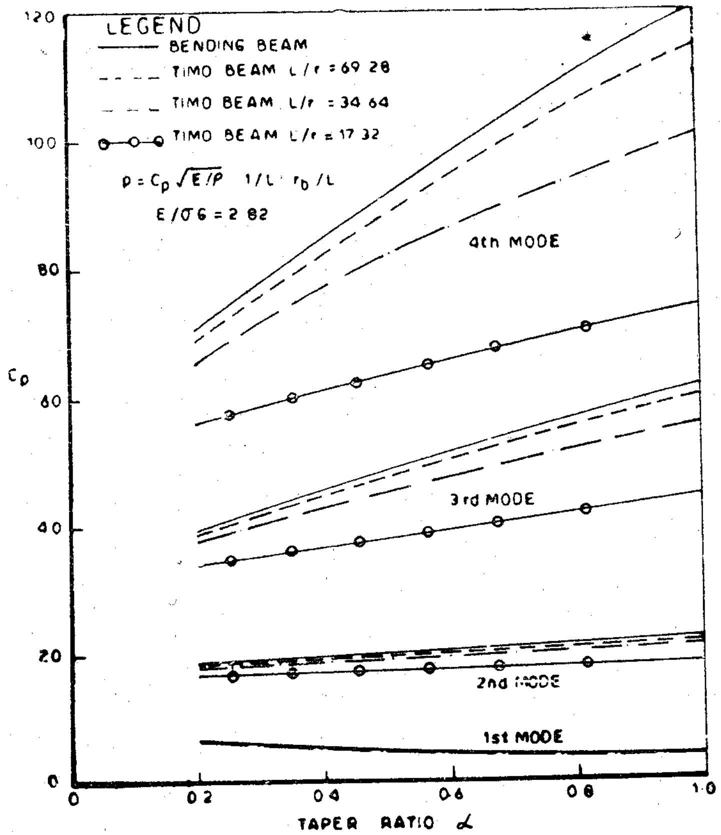

Fig. 10. Frequency vs Taper Ratio for Various Modes and Slenderness Ratios

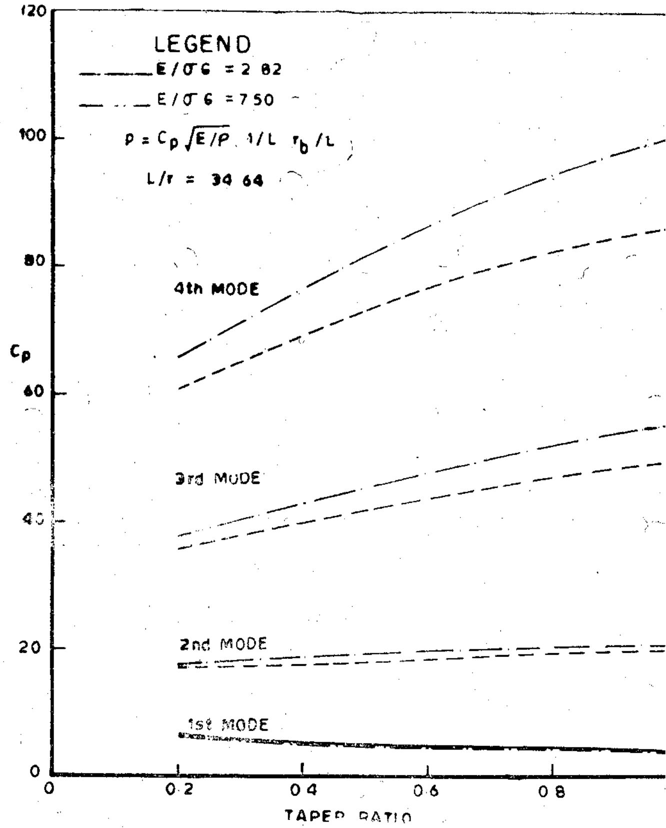

Fig. 11. Frequency vs Taper Ratio for Various Modes and E/σG Ratios

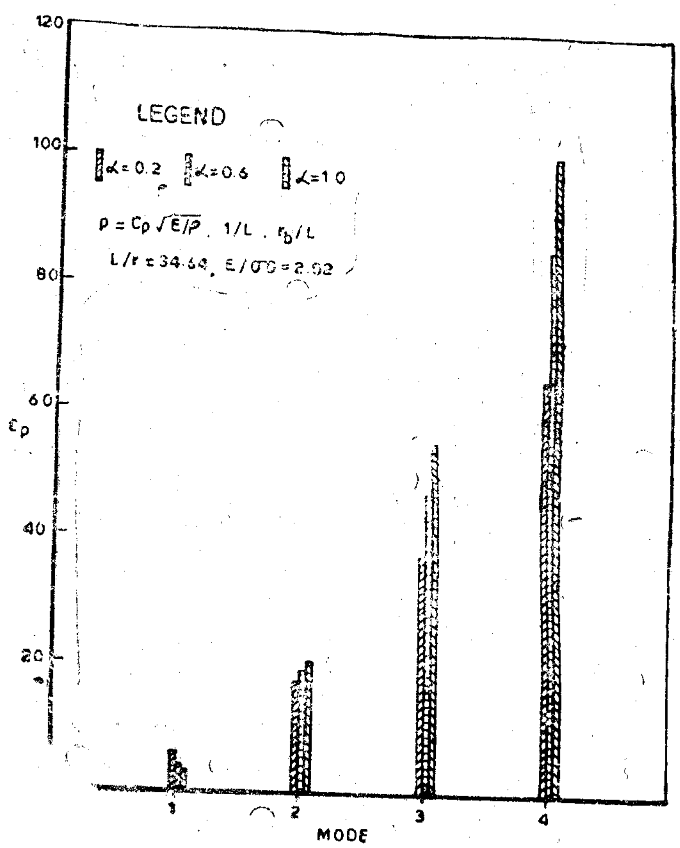

Fig. 12. Effect of Mode of Vibration on Frequency. (For Different Taper Ratios of Timoshenko Beam)

Figures 10 to 12 show the influence of taper ratio on the frequency. In the first mode, frequency decreases with increase in taper ratio whereas in other modes it is vice versa. In higher modes, frequency increases rapidly with increase in taper ratio.

Frequency decreases with increase in the value of parameter $E/\sigma G$. That is, a solid section, having the same slenderness ratio, length and taper and made of same material as a hollow section would have a larger frequency than the hollow section.

In summary, if higher modes of vibration are to be considered and the beam is stout, shear and rotatory inertia deformations are to be considered in addition to that of bending.

10—Acknowledgement

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Notations

A — area of cross-section
a — a constant. 'ka' is the radius of gyration at the free end of the cantilever beem.
a(t) — acceleration of base
b — a constant. 'k(a+b)' is the radius of gyration at the fixed end of the cantilever beam.
C — mode factor
E — modulus of elasticity
G — modulus of rigidity
I — moment of inertia of cross-section
k — a constant. proportional to radius of gyration.
L — length of beam
M — bending moment
m — mass
n — number of masses
p — natural frequency of vibration
r — radius of gyration
t — time
V — shear force
w — force
x — distance along the beam
Y — mode shape
y — total deflection
y_b — deflection due to bending
y_s — deflection due to shear
Z — relative displacement between the beam element and base
α — Taper ratio $= \frac{a}{a+b}$
β — frequency parameter. Has a unit of L^{-1}
γ — frequency parameter. Proportianal to p^2
ϵ — error in γ
θ — phase angle
ν — poissoin's ratio
ξ — normal coordinate
ρ — mass density
σ — ratio of average shear stress on a section to the product of shear modulus and the angle of shear at the neutral axis
τ — a parameter of time
ϕ — mode shape corresponding to bending
ψ — mode shape corresponding to shear