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UTILIZING THE STRAIN ENERGY METHOD TO STUDY A CIRCULAR PLATE WITH A CONCENTRATED LOAD RESTING ON WINKLER FOUNDATION

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ABSTRACT

This study uses the strain energy approach to present a fundamental analytical expression for the moments and deflection of a circular elastic plate resting on a Winkler foundation. Parametric analysis is performed to assess the location of the plate liftoff and how the deflection profiles changed, with a view to contrasting the pattern of variation reported in the literature using an approximation solution method. It comes to light that the plate's point's radial distance of lift-off decreases more for lower values of the foundation soil's subgrade coefficients. Furthermore, it has been found that variations in the soil's subgrade conditions can cause a considerable fluctuation at any radial distance in the plate's deflection, particularly at lower values of the modulus of subgrade reaction. The present formulation has exceptional performance, precision, and applicability.

KEYWORDS: Winkler foundation; Modulus of subgrade reaction; Circular flexible plate; Plate lift-off

INTRODUCTION

On elastic foundations, circular plates have been given a lot of attention because of their widespread use in structural, mechanical, and civil engineering, including water tanks, airport runways, railway tracks, and pavements. Given the complex soil-structure interaction problem that these kinds of constructions provide, it is challenging to understand how they transfer vertical or horizontal forces to the foundation.

(Winkler 1867) developed a very basic model based on the supposition that each foundation point's reaction forces per unit area correspond to the foundation's actual deflection. The majority of analytical and numerical methods are used to analyze circular plates resting on elastic foundations under a centrally concentrated load (Olson and Lindberg 1970; Eisenberger and Clastornik 1987b, a; Celep 1988; Melerski 1989; Celep and Turhan 1990; Guler and Celep 1995; Chandrashekhara and Antony 1997; Buczkowski and Torbacki 2001; Güler and Guler 2004; Rao and Rao 2013). (Dutta et al. 2023, 2024) developed a higher-order finite element method to evaluate circular rafts on elastic foundations. (Shukla et al. 2011) evaluate a thin circular plate on the Pasternak foundation using the strain energy technique in order to give a straightforward analytical equation for plate deflection and investigate plate lift-off.

For the study of circular plates supported by elastic foundations and loaded axisymmetrically, (Utku et al. 2000) provide a “mixed formulation” of the stiffness and flexibility parameters obtained from the classic plate for annular plate. (El-Zafrany and Fadhil 1996) used the boundary element approach to evaluate thin plates resting on a two-parameter elastic base. (Karaşin et al. 2015) developed finite grid solutions on elastic foundations for circular plates. (Yue et al. 2020) utilizing two modified Vlasov models, consider the bending study of thin circular plates held up by flexible bases.

Because they lack straightforward plate response techniques and necessitate extensive mathematics, two-parameter and continuum-based models are rarely utilized. However, Finite element modelling in three dimensions is expensive and time-consuming. Therefore, the main purpose of this study is to establish an analytical formula for the deflection and moments of a circular elastic plate subjected to a centrally concentrated load resting on an elastic foundation. To attain this objective, the elastic foundation is described using the Winkler model. By employing the strain energy technique in order to produce a simple analytical formula for the plate deflection, as well as moments are also obtained. The strain energy approach is used to solve the mechanical response of plates and foundations. Numerical examples are used to do the parametric analysis. The influence of the modulus of subgrade reaction and Poisson's ratio of plate material on the deformation and internal forces of foundation plates is discussed in detail, as well as the lift-off of the plate, where applicable.

In order to produce a straightforward analytical formula for the plate deflection and moments and to analyse the plate lift-off, the present study investigates at a circular flexible plate that is laying on a Winkler foundation that use the strain energy technique. Keeping more terms of the series given in equation (1) leads to an improved approximation. The present study (PS) provides various results. Practitioners and design engineers may find the element appealing and beneficial due to its simplicity of formulation, ability to produce precise moment values, cost-effectiveness, and possibility for requiring less computational work, resources, and time. As a result, it serves as a resource for aspiring professionals and design engineers in this area.

PROCEDURES

A somewhat small model is the Winkler Foundation (Figure 1). Practicing engineers used the Winkler model for routine work because of its simplicity. In this model, it is assumed that reaction forces per unit area are proportional to foundation deflection. The combination of discrete, equal, independent, linearly elastic springs known as the subgrade response modulus, or K, controls the foundation's vertical deformation properties. The Winkler model states that q = Kw, where w represents the foundation surface's deflection under pressure, q. In (Dutta et al. 2021, 2022), how to estimate the foundation parameter modulus of the subgrade response is described.

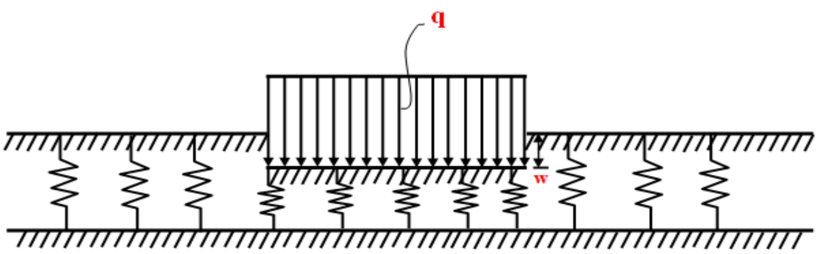

Fig. 1 Winkler Foundation

1 Plate Deflection Analytically Derived

One can presume that the plate bending solution has the form of a series:

$$w = c_0 + c_2 r^2 + c_4 r^4 + c_6 r^6 + c_8 r^8 \ldots\ldots\ldots. + c_{2n} r^{2n} \tag{1}$$

because of the symmetry in the situation (Ugural 1999)

Hence, provided that the system's total potential energy, Π, is minimal in stable equilibrium, the unknown coefficients c_0, c_2, c_4, …….c_{2n}, can be found.

Where unknown coefficients c_0, c_2, c_4, …….c_{2n}, can be ascertained provided that the system's total potential energy Π, is minimal in a stable equilibrium.

If we retain only the first five terms of equation (1), then deflection could then take the form of

$$w = c_0 + c_2 r^2 + c_4 r^4 + c_6 r^6 + c_8 r^8 \tag{2}$$

The total potential energy of the soil-structure system could be equivalent to

$$\Pi = \Pi_p + \Pi_s - V \tag{3}$$

Where the plate's total strain energy is represented by Π_p. Where Π_s is the soil's stored strain energy and V is the potential energy of the external loads. In this concept, the loads operate only laterally on the plate domain and can be concentrated in the center. When a problem is axis-symmetric, all quantities are functions of only r and z. There is no variation in any quantity with respect to the circumferential coordinate θ. Not a single point moves in the direction of θ. As a result, the axisymmetric circular plate's strain energy is provided by

$$\Pi_p = \frac{D}{2}\int_0^{2\pi}\int_0^a\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\frac{1}{r}\frac{\partial w}{\partial r}\right]rdrd\theta = \pi D\int_0^a\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\frac{1}{r}\frac{\partial w}{\partial r}\right]rdr \tag{4}$$

Where $D = \frac{Eh^3}{12(1-v^2)}$ E stands for the plate's elastic modulus, h for its thickness, and v for its Poisson's ratio.

The Winkler Foundation provides the following response function:

q = Kw, where q is the distributed reaction from the foundation brought on by the imposed concentrated load and K is the Winkler springs subgrade reaction modulus. The circular plate is an axisymmetric example that is subject to an accumulated load P at its middle O.

The strain energy Π_s resulting from the Winkler foundation's deformation can be computed as follows:

$$\Pi_s = \int_0^{2\pi}\int_0^a \frac{1}{2}qwrdrd\theta = \frac{1}{2}\int_0^{2\pi}\int_0^a Kw^2rdrd\theta = \pi\int_0^a Kw^2rdr \tag{5}$$

The following quantity of work (V) is produced by the load acting at the circular plate's center: $V = F\times w_{r=0}$.

When a concentrated load F is applied to the circular elastic plate foundation of the soil-plate system at its center O, the total potential energy of the system is given by $\Pi = U_{Plate} + U_{soil} - V$.

At this point, the soil-plate system's potential energy equals

$$\Pi = \frac{D}{2}\int_0^{2\pi}\int_0^a\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\frac{1}{r}\frac{\partial w}{\partial r}\right]rdrd\theta + \int_0^{2\pi}\int_0^a\frac{1}{2}qwrdrd\theta - F\times w_{r=0} \tag{6}$$

$$\Pi = \pi\left[D\left\{\frac{64{c_8}^2}{7}(7\nu+25)a^{14} + 96c_6c_8\nu(3+\nu)a^{12} + \frac{4}{5}(45{c_6}^2\nu + 117{c_6}^2 + 176c_4c_8 + 80c_4c_8\nu)a^{10} + 16(2c_2c_8 + 6c_4c_6 + 2c_2c_8\nu + 3c_4c_6\nu)a^8 + \frac{8}{3}(3{c_4}^2\nu + 10{c_4}^2 + 9c_2c_6 + 9c_2c_6\nu)a^6 + 16c_2c_4(1+\nu)a^4 + 4{c_2}^2(\nu+1)a^2\right\} + K\left\{\frac{a^{18}{c_8}^2}{18} + \frac{a^{16}c_6c_8}{8} + \frac{a^{14}}{14}({c_6}^2 + 2c_4c_8) + \frac{a^{12}}{6}(c_2c_4 + c_4c_6) + \frac{a^{10}}{10}({c_4}^2 + 2c_0c_8 + 2c_2c_6) + \frac{a^8}{4}(c_0c_6 + c_2c_4) + \frac{a^6}{6}({c_2}^2 + 2c_0c_4) + \frac{a^4c_0c_2}{2} + \frac{a^2{c_0}^2}{2}\right\}\right] - c_0F \tag{7}$$

For a stable equilibrium, potential energy should be at its lowest, allowing the following application of the condition:

$$\frac{\partial\Pi}{\partial c_0} = 0; \frac{\partial\Pi}{\partial c_2} = 0; \frac{\partial\Pi}{\partial c_4} = 0; \frac{\partial\Pi}{\partial c_6} = 0 \text{ and } \frac{\partial\Pi}{\partial c_8} = 0$$

The coefficients for the five simultaneous equations are c_0, c_2, c_4, c_6 and c_8.

Substituting the coefficients c_0, c_2, c_4, c_6 and $c_8$into equation (2) yields the plate deflection solution.

$$M_r = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r}\right) \tag{8}$$

$$M_t = -D\left(\frac{1}{r}\frac{\partial^2 w}{\partial r^2} + \nu\frac{\partial w}{\partial r}\right) \tag{9}$$

RESULTS AND DISCUSSIONS

The (Inc. 2011) environment was used to write a code for meshing three-noded plates and four-noded plates. Next, open (Systems 2014) and place the joint coordinates and element incidences in the appropriate Staad editor from Matlab. Next, select the thickness and material property of each plate element material from the property tab and use the Staad editor to edit the constants Poisson's ratio and elastic modulus. Finally, select the Supports tab, Y subgrade, and print the plate. Choose Joint Load 1 FY under the Load and Definition tab, and then Perform Analysis and Print Joint Displacement List 1 and Print Element Joint Stresses List 1.

To facilitate results comparison, the following Non-dimensional (ND) parameters are defined:

1. ND deflection $\overline{w} = \frac{1600 w \pi D}{F r^2}$ and ND moment $\overline{M} = M/F$ for concentrated load, F.
2. Foundation parameter, G* = Kr^4/D and $P^* = \frac{Fr}{D}$, $w^* = \frac{w}{r}$and $r_b = \frac{d}{r}$where d is the radial distance between the plate's center and the starting point of the plate's lift-off.

Validation

An example problem has been chosen: a circular plate with a radius of 1 m and a thickness of 0.05 m. Young's modulus and Poisson's ratio of the plate material are 2.1× 10^5 MN/m^2 and 0.3 on the Winkler foundation, which is considered to be validations with the software (Systems 2014) for the free (F) plate under a central concentrated load of 3 MN and the modulus of subgrade reaction of 10 MN/m^3. The comparison of the PS solution with the FE solution (Systems 2014) is shown in Table 1, and Table 2 presents the solution's improvement by incorporating additional terms. The results with respect to the (Systems 2014) can be accepted as excellent.

Table 1: Maximum and minimum deflection and a radial bending moment of a free plate with a central concentrated load

Response	PS	FE solution (Systems 2014)	Difference (%)
Maximum deflection (mm)	117.001	120.8970	3.22
Minimum deflection (mm)	82.519	82.9080	0.47
Maximum radial bending moment (kN-m/m)	477.85	466.66	2.40

Table 2: Maximum and minimum deflection of a free plate with a central concentrated load

Response	PS (Solution using five – terms)	Solution using two - terms	FE solution (Systems 2014)
Maximum deflection (mm)	117.001	104.7342	120.8970
Minimum deflection (mm)	82.519	86.2517	82.9080

A circular plate with a diameter of 1 m and a thickness of 0.06 m is being used on a Winkler foundation to test the free (F) plate when subjected to a modulus of subgrade response of 20 MN/m^3 and a central concentrated load of 4 MN. The material of the plate also has a Poisson's ratio of 0.25 and a Young's modulus of 2.1 × 10^5 MN/m^2, respectively. Furthermore, Table 3 presents a comparison between the PS solution and the reference offered by (Systems 2014) and (Dutta et al. 2023). As a result, it is possible to consider the results pertaining to the reference to have excellent agreement.

Table 3: Maximum and minimum deflection of a free plate with a concentrated load in the center

Response	PS	FE solution (Systems 2014)	(Dutta et al. 2023)
Maximum deflection (mm)	80.883	83.967	81.892
Minimum deflection (mm)	53.183	53.451	53.112

Studies Using Parametric Data

The findings presented in Figures 2 - 6 illustrate the nature of parameter variation with different soil and plate parameters for the parametric analysis of the parameters G*, P*, and v.

The results may be practically used for concrete plate (Poisson's ratio 0.10 – 0.20), steel plate (Poisson's ratio 0.27 – 0.30), as well as aluminum plate, rock, etc. (Poisson's ratio greater than 0.33).

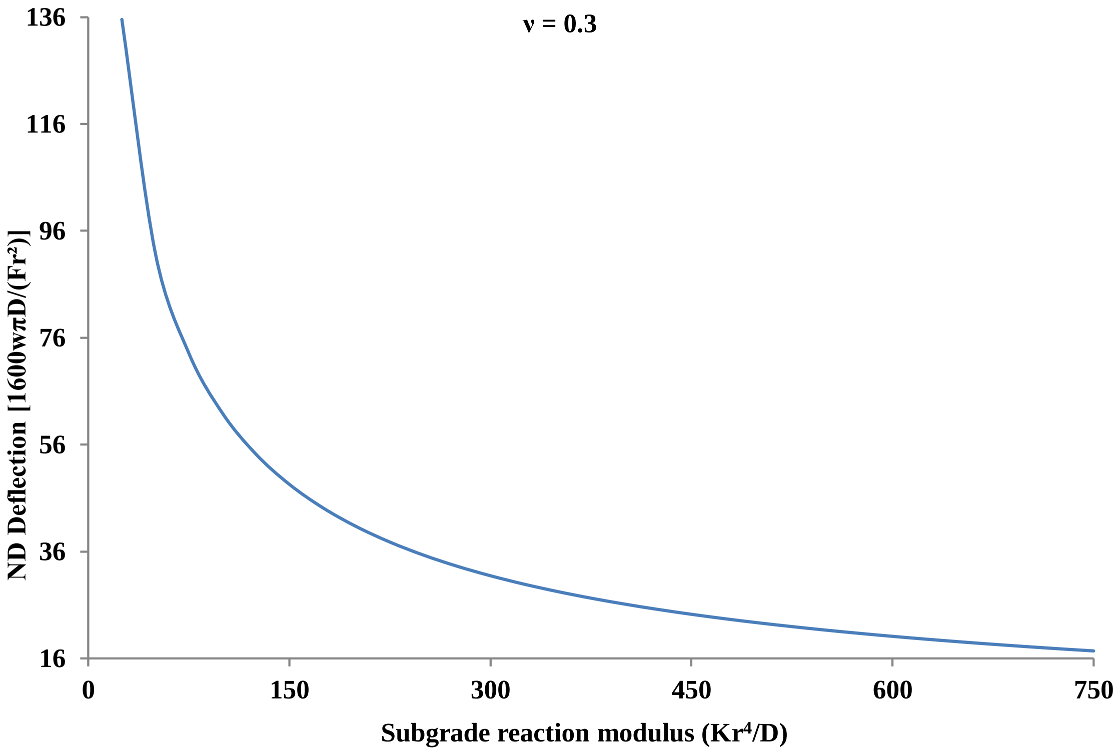

Fig. 2 ND Central Deflecting the free circular plate with a centered load

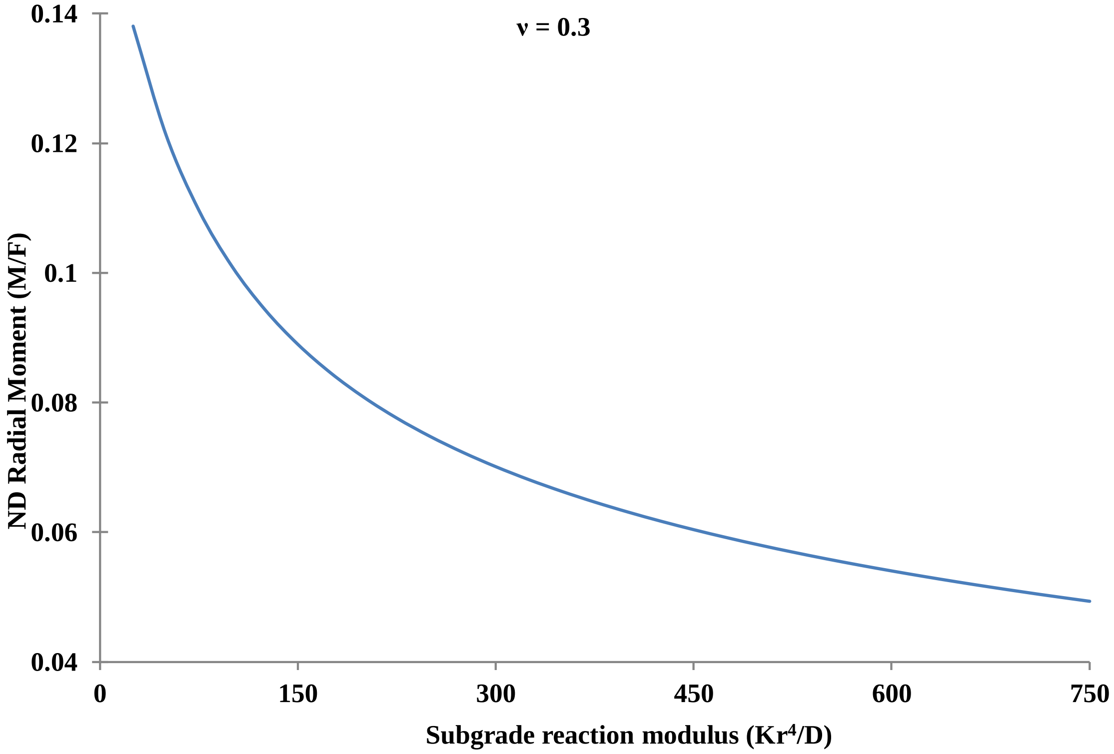

Fig. 3 The free circular plate's ND central moment with a focused load

Figures 2 and 3 illustrate how the ND central displacement parameter and the ND central bending moment parameter vary with respect to the non-dimensional foundation parameter. The value of $(\overline{w})$ decreases nonlinearly with an increase in G*, with the rate of reduction of $(\overline{w})$ being larger for lower values of G* (let's say less than 200). Figures 2 and 3 indicate that the displacement parameter and bending moment parameter fall as the foundation parameter, G*, increases and also that the $\overline{w}$ and $\overline{M}$ tend to stabilize as the foundation parameter G* increases.

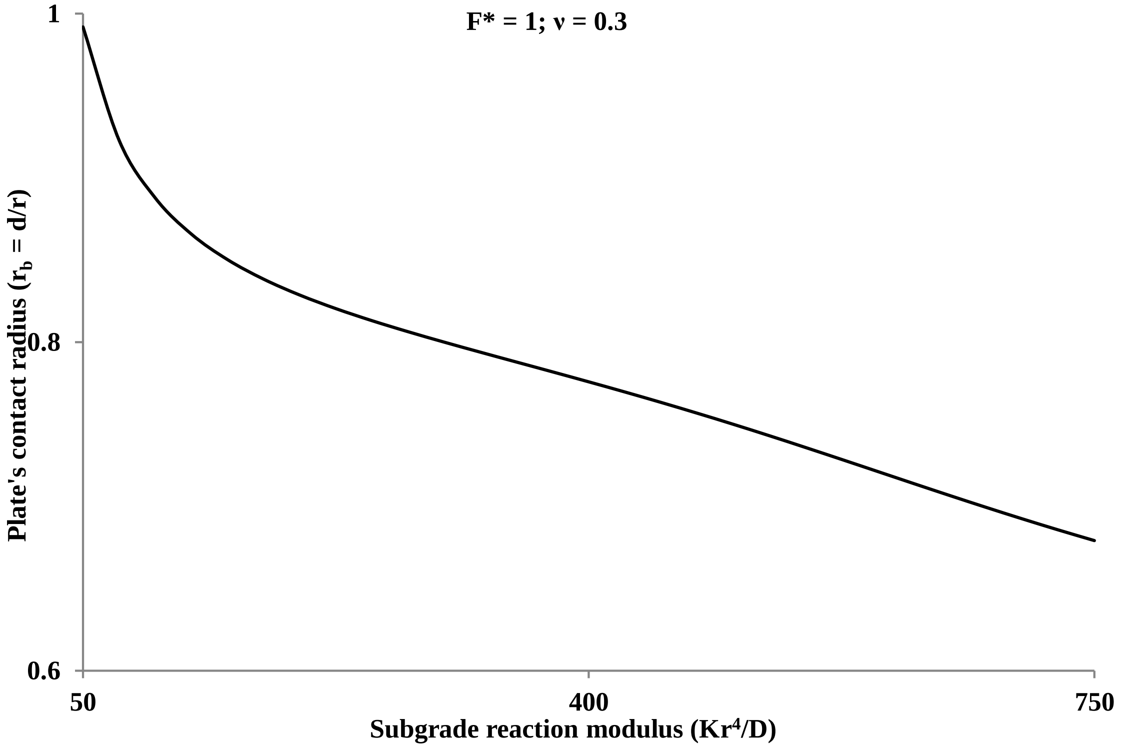

Fig. 4 The relationship between the non-dimensional subgrade reaction modulus (G*) and the fluctuation of the plate's contact radius ($r_b = d/r$)

The relationship between the non-dimensional subgrade response modulus (G*) values and the non-dimensional contact radius (r_b) is depicted in Figure 4. With an increase in G*, the value of r_b declines nonlinearly, with a greater rate of reduction for smaller values of G*. Therefore, the relationship between r_b and G* is nonlinear.

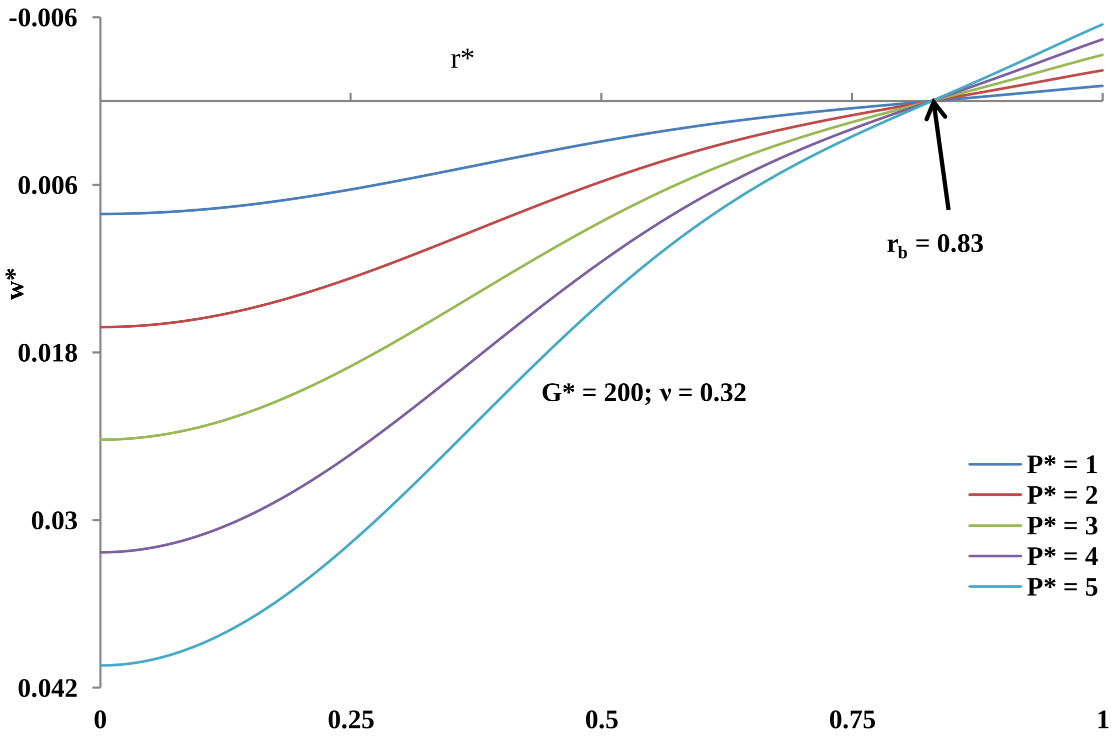

Fig. 5 The deflection curves of the plate for different values of the non-dimensional concentrated load (P*)

An illustration of the non-dimensional behavior of a circular plate's deflection as a function of its distance (r_b) from the center under various concentrated load levels (P* = Fr/D) can be found in Figure 5. It should be noted that for a range of P* values, the plate lift-off develops at the contact radius, r_b = 0.83.

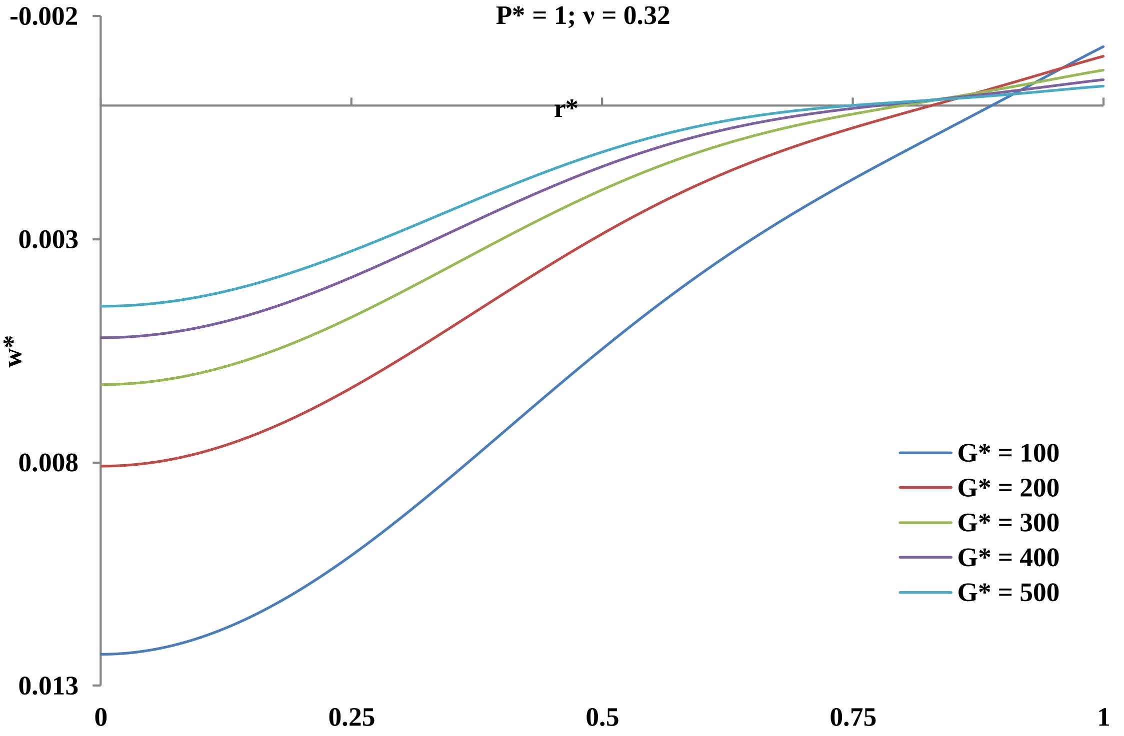

Fig. 6 Profiles of the plate's deflection for different non-dimensional subgrade response (G*) values

The relationship between the radial distance (r*) and the plate deflection (w*) for a range of values of the modulus of subgrade response (G*) is displayed in Figure 6. The relationship in terms of dimensions is not depicted. We see that this is true for any value of G*.

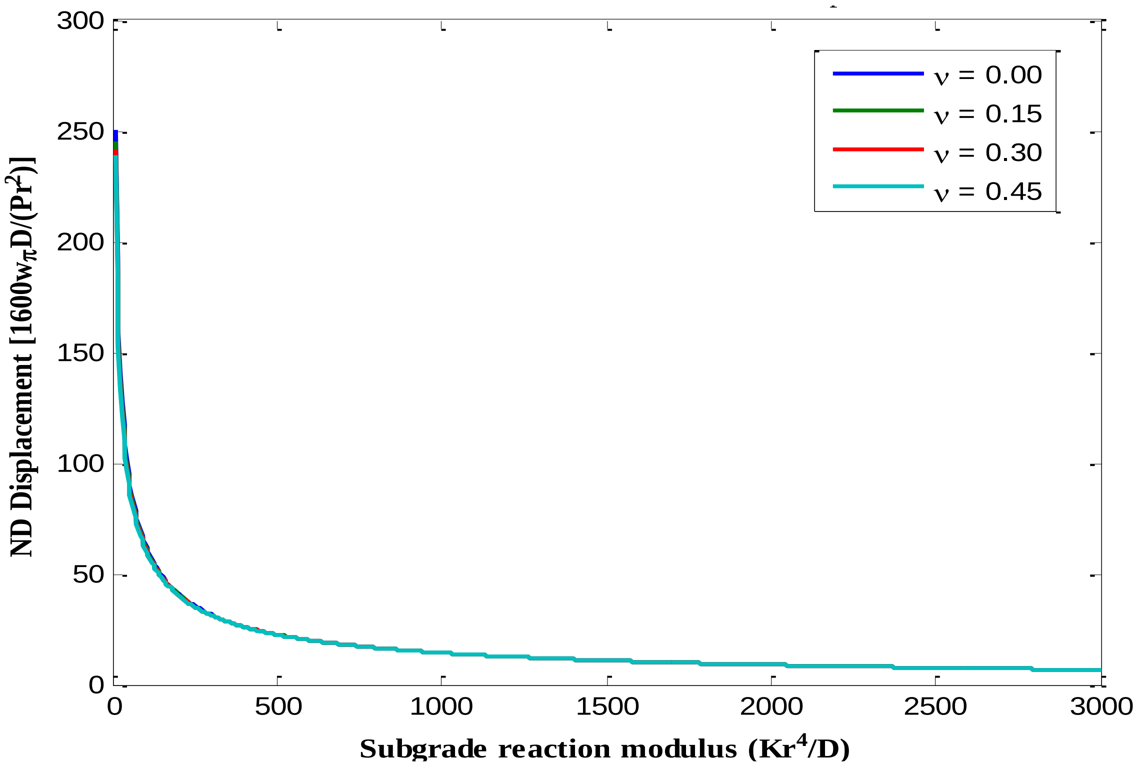

Fig. 7 ND central Deflection versus ND foundation parameter

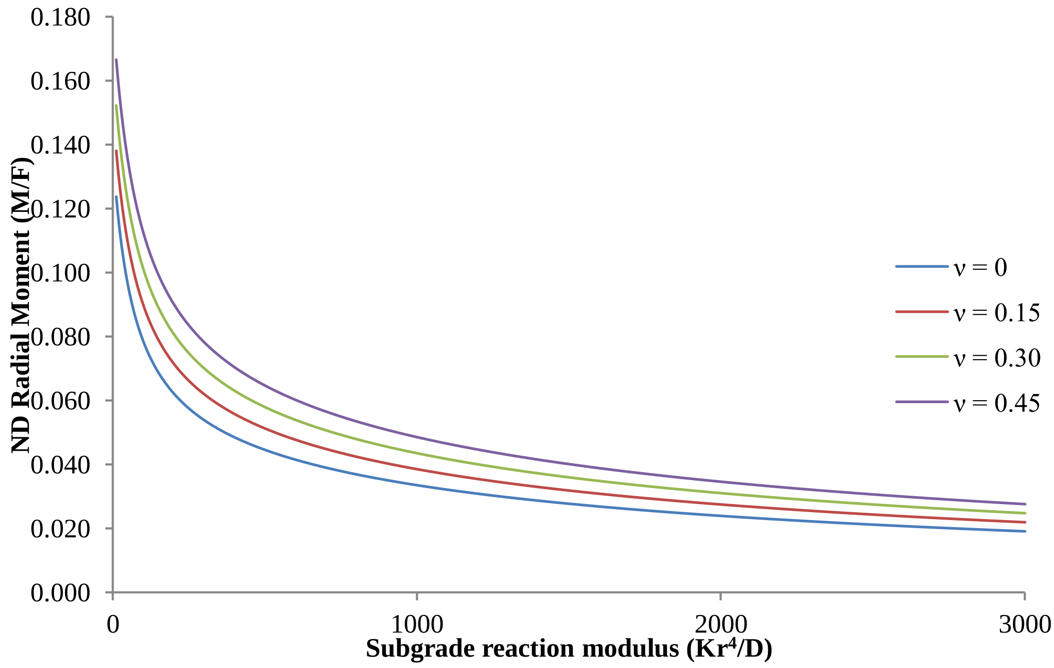

Fig. 8 ND Radial Moment versus ND foundation parameter

Figures 7 and 8 present the values of normalized plate deflection and radial moment versus ND foundation parameter for a wide range of plate Poisson's ratio and ND foundation parameter.

Figure 7 shows the relationship between the plate deflection (w*) and the ND subgrade response (G*) for a number of different values of the Poisson's ratio of plate material (ν). It does not show the relationship in terms of dimensions. It has been observed that this holds for any value of ν. The value of ($\overline{w}$) decreases nonlinearly with an increase in G*, with the rate of reduction of ($\overline{w}$) being larger for lower values of G* (let's say less than 200). Figure 7 indicates that the displacement parameter falls as the foundation parameter, G*, increases and also that the $\overline{w}$ tend to stabilize as the foundation parameter G* increases.

Figure 8 shows the relationship between the radial moment$\overline{M}$and the ND subgrade response (G*) for a number of different values of the Poisson's ratio of plate material (ν). It does not show the relationship in terms of dimensions. It has been observed that this holds for any value of ν. The value of $\overline{M}$ decreases nonlinearly with an increase in G*, with the rate of reduction of $\overline{M}$ being larger for lower values of G* (let's say less than 200). Figure 8 indicates that the ND Radial Moment parameter falls as the foundation parameter, G*, increases and also that the $\overline{M}$ tends to stabilize as the foundation parameter G* increases.

The effects of soil stiffness on the flexibility, displacement, and moment variation of a plate subjected to a concentrated load at the centre were investigated parametrically using the Formulation developed in this study. Typical conditions are depicted with E_p/E_s = 10,000 (i.e., E_p = 200 GPa and E_s = 20 MPa) for a steel plate and with E_p/E_s = 800 (i.e., E_p = 25 GPa and E_s = 31.25 MPa) for a concrete plate; however, the stiffness ratios E_p/E_s of the plate to soil vary depending on soil stiffness. Poisson's ratio (ν_s) 0.17 of the soil was considered in all future analyses, with steel and concrete assumed to have Poisson's ratios of 0.31 and 0.16, respectively. Figure 9 show the parametric results of the study.

Effect of Soil Stiffness or Relative Stiffness

Further research was done on the impact of soil stiffness on plate flexibility for both concrete and steel tanks. For a circular column of a structure carrying a load of 500 kN, a radius of isolated footing (r) of 1.5 m and for concrete, h (= 75 mm), E_p (= 25 GPa), and ν_p (= 0.16) were held constant while E_p/E_s was changed from 800 to 8000. E_p/E_s = 800 (i.e., E_s = 31.25 MPa), for concrete, denotes stiff ground. On the other hand, weak soils, like incredibly soft clays, are represented by E_p/E_s = 8000 or E_s = 3.125 MPa. Conversely, using steel tanks, the stiffness ratio E_p/E_s was changed from 10000 to 60000 while maintaining constants for h (= 30 mm), E_p (= 200 GPa), and ν_p (= 0.31). Regarding steel tanks, stiff geomaterials are represented by E_p/E_s = 10000 (i.e., E_s = 20 MPa), while delicate soils are represented by E_p/E_s = 60000 (i.e., E_s = 3.33 MPa). Depth of influence zone H = 2×(2r) = 6 m based on Boussinesq's method. The values of the parameter $K = \frac{E_s(1-v_s)}{H(1+v_s)(1-2v_s)}$ can be calculated using the (Selvaduari 1979) technique.

Figure 9 shows the relationship between w(r)/w(0) and r/r_p for different E_p/E_s values for steel tanks Figure 9(a) and concrete Figure 9(b). For both steel and concrete plates with B = 3 m, as illustrated in Figures 9(a) and (b), w(r)/w(0) declines with rising E_p/E_s, suggesting that plates become more flexible as the founding soils are stiffer. For the 3 m diameter steel plate with h = 30 mm, the edge-to-center settlement ratios [see Figure 9(a)] are around 0.061 and 0.611, respectively, founded on soils with E_p/E_s = 10000 (stiff) and 60000 (incredibly soft) and concrete slab the edge-to-center settlement ratios [see Figure 9(b)] are around 0.082 and 0.632, respectively, founded on soils with E_p/E_s = 800 (stiff) and 4800 (incredibly soft). Remember that a higher differential settlement is not always indicated by a more significant value of w(r)/w(0). In extremely stiff soils, w(r)/w(0) may be significant, yet the differential settlement $[w(0) - w(r_p)]$ is small because of the small settlement magnitude. Further evidence that the larger plate's flexibility is less dependent on the founding soil's stiffness.

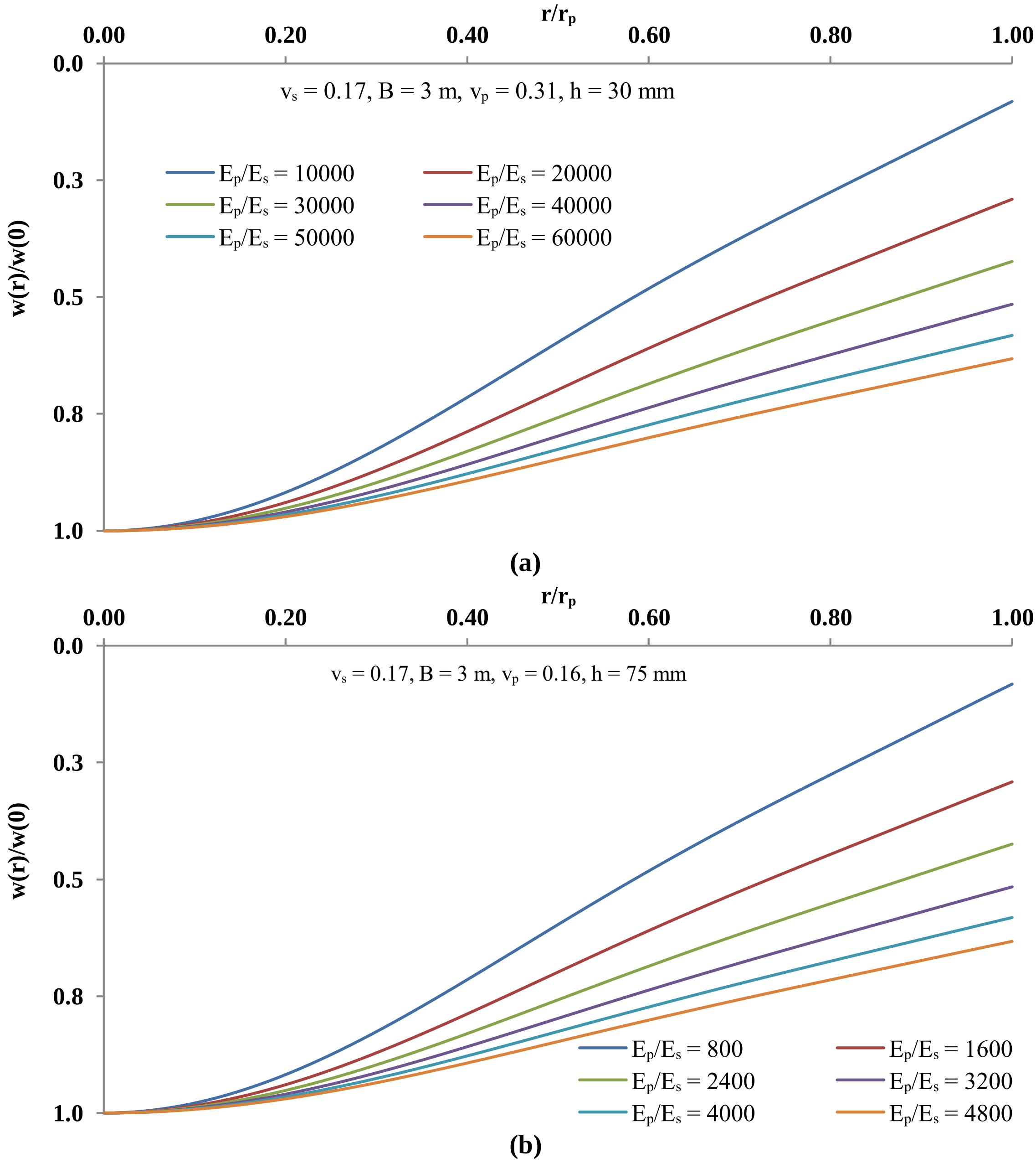

Fig. 9 Normalized plate deflection versus normalized radial distance with various soil stiffness for steel tanks (a) and concrete tanks (b)

As the sub-soil elastic modulus increases the plate's central displacement and bending moment always decrease. It is also observed that the sub-soil Poisson's ratio has a negligible effect on displacement and bending moment.

Now considered a practical problem as a circular column of a structure carrying a load of 1500 kN, a radius of isolated footing (r) of 1.5 m, and a thickness of 75 mm, the plate's material also has a Young's modulus (M_{35} concrete) and a Poisson's ratio of 29580.4 MN/m² and 0.2, respectively. Soil properties are E_s = 20 MPa, ν_s = 0.17, density, and H = 2×(2r) = 6 m based on Boussinesq's method.

The values of the parameter K can be calculated using the (Selvaduari 1979) technique.

First, the ND Foundation parameters are calculated as G* = 16.74 and D = 1083266.6 N-m from Figures 7(c) and 7(e) $\overline{w}_{(\nu=0.15)} = 175.5$ and $\overline{w}_{(\nu=0.3)} = 171.5$ so $\overline{w}_{(\nu=0.2)} = 175.5 - \frac{4\times0.05}{0.15} = 174.2$. Therefore, the central displacement of the plate is calculated as $w = \frac{\overline{w}Pr^2}{1600\pi D} = 108\ \text{mm}$ and the displacement from (Systems 2014)is 113.8 mm.

In all the above cases, the reference-related outcomes can be deemed satisfactory.

CONCLUSIONS

The findings that FE solution and other researchers have gotten verify the accuracy and efficacy of the current formulation for various foundation parameters. Additionally, a parametric research is carried out to elucidate the impacts of modifications in various parameters. The current elements produce a somewhat accurate forecast regarding the deflection and stress values and function well for concentrated loads on solid circular-type plates sitting on Winkler foundations, according to a range of numerical studies.

- ✓ The numerical results for intense solid round-type plate loading states won't only show the current elements' value. However, they will also serve as a useful resource for upcoming experts in this area.
- ✓ The element's ease of detailing, ability to provide excellent moment values, cost-effectiveness, and ability to need less computational effort, resources, and time may be the reason for its attraction and usefulness to experts and design engineers.
- ✓ The results of this study are consistent with the exact and numerical conclusions documented in the literature.
- ✓ In this work, an analytical equation for the deflection of a thin, circular elastic plate subjected to a concentrated load while resting on the Winkler foundation is developed using the strain energy technique.
- ✓ For a certain value of the focused load, the plate's deflection decreases as the radial distance increases.
- ✓ When the modulus of subgrade response is smaller, variations in the subgrade lead to significant variations in the amount of plate bending at a given radial distance.

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