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FREE VIBRATION ANALYSIS OF BEAMS ON ELASTIC FOUNDATION USING QUINTIC DISPLACEMENT FUNCTIONS

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ABSTRACT

Free vibration analyses of beams on elastic foundation are very common in civil and mechanical engineering. Such types of structures are exposed to dynamic loads is a complex soil-structure interaction problem. In the present study for safe and economical design of such structures, a workable approach for free vibration analysis of beams on Winkler foundation using first-order continuity (C^1) two degree of freedom (DOF) per node three nodded beam based on Euler-Bernoulli beam theory (EBBT) is attempted. A Matlab code is developed for the present formulation. The results, thus obtained, are compared with similar studies done by other researchers as well as with exact solution where applicable, which show very good conformity and a maximum difference 0.24% with exact solution for mode eight. It is concluded that the present formulation has rapid convergence regardless of boundary conditions, depth to length ratio of beam and modulus of sub-grade reaction. It performs extremely well for thin beams in terms of ease and consistency and gives a very accurate result with only few elements and within few seconds.

KEYWORDS: Free Vibration, Winkler Foundation, Modulus of Sub-Grade Reaction, Quintic Beams, First Order (C^1) Continuity

INTRODUCTION

There are many foundation models such as the Winkler model, the two-parameter Pasternak model, and Vlasov model, etc. It is very important to choose a more practical foundation models and simpler methods for the safe and economical design of such complex soil-structure interaction problem. The majority of the problems cannot be solved by the theoretical approach. In solving this type of problem; one can use numerical techniques like the finite element method.

Euler-Bernoulli proposed a highly basic beam model based on the assumptions that "plane portions remain flat" after bending and that the deformed beam slope is modest. The Euler-Bernoulli beam theory is a simplified version of the linear theory of elasticity that can be used to estimate beam load carrying and deflection characteristics. Exact deflections and slopes at nodes can be obtained using classic beam elements approximated by a cubic polynomial with first order (C^1) continuity. Engineers are interested in the second and third derivatives of the solution, which are the local moment and shear force; traditional beam elements produce poor design results until a large number of elements are used. Many accurate beam solutions for the most common load and support conditions are polynomials of third, fourth, or fifth degree. As a result, it is evident that the accuracy with which the deflection of the foundation structure is determined is important to the behavior of the beam on elastic foundation. Thus, a fifth degree polynomial will generally give a very accurate or exact solution of beam on elastic foundation with only few elements. By increasing the number of nodes and/or the continuity level at each node, the Hermite family of interpolation polynomials can be raised in order. It is chosen to use a three-nodded beam element with two degrees of freedom per node. The computational efficiency has substantially enhanced, which makes post-processing much faster.

To developed exact moment diagram and shear diagrams due to concentrated moments and loads the finite element meshing should be set up so that every point source occurs at an interface node between elements. Akin [1] discuss in his paper about why a quintic beam was not chosen for a two-node second-order (C^2) Hermite interpolation.

Free vibration analysis of beams resting on the Winkler foundation is important for the structural design of beam. The mechanical modeling of beam-subsoil interaction problem is mathematical quite a complex phenomenon and the response of sub-grade is depending upon many factors.

Hetenyi [2] discuss in detail in his paper the beam solution on Winkler foundations. The key advantage of the commonly used Winkler model is its simplicity and the only input needed is the sub-grade reaction module K. Kim and Kim [3] regarded the vibration of beams with boundary requirements of general restriction. Friswell et al. [4] presented beam vibration, using the finite element principle for a nonlocal viscoelastic foundation model. The dynamic reaction of the Euler-Bernoulli beam to a focused moving force on linear and nonlinear viscoelastic Winkler foundations was considered by Senalpet al. [5]. Hizal, and Çatal [6] developed the functional analysis on modified Vlasov foundations of axially loaded beams. Thambiratnam and Zhuge [7] have formulated a basic model to study the dynamic behavior of beams on a flexible foundation under moving load conditions. Coskunet al. [8] studies in details the free vibration analysis of non-uniform Euler beams on elastic foundation using the approach of domain decomposition. Robinson [9] in his paper discusses in details the principles of seismic isolation and application of isolation systems to structures in Zealand.

Rahbar-Ranji and Shahbaztabar [10] uses Legendre polynomials and the Rayleight-Ritz method to analyze the free vibration analysis of beams on the Pasternak foundation. Sapountzakis et al. [11], presented nonlinear dynamic analysis of partly supported Timoshenko beam-columns on the tensionless Winkler foundation. Kacar et al. [12] studies the free vibration of beams on variable Winkler elastic foundation using differential transformation. For Beam on linearly varying sub-grade modulus, Chen et al. [13] developed an improved solutions approach using quintic displacement functions. Dutta et al. [14] developed finite element formulation for three-nodded beams on Pasternak type foundation using quintic displacement functions. Dutta and Mandal [15, 16] developed finite element formulation for dynamic analysis of thin beam and deep beam on Vlasov foundation.

Practitioners rarely employ the two-parameter and continuum-based models because they involve extensive mathematics and do not provide straightforward techniques for free vibration analysis of beams on elastic foundation. The finite element method, in its three-dimensional form, has the potential to produce realistic results free vibration analysis of beams, if appropriate soil constitutive relationships are used and appropriate elements and domains for the soil and beam are chosen. But it is a time-consuming and expensive.

In the present study, adopting a three nodded beam element having two DOF per node based on EBBT for free vibration analysis of beams on elastic foundation is attempted. It overcomes inconsistency of two nodded beam elements having two DOF per node. To model the elastic foundation modulus of sub-grade reaction is the only input. It has rapid convergence and performs exceptionally well for thin beam and saves computer memory, resulting in lower computational cost, time and resources.

A Matlab [17] code is written based on the present formulation. Convergence study is carried out and results are validated with available literature, which shows very good conformity. Convergence rate, accuracy, and applicability of the present formulation for free vibration analysis of beams on Winkler foundation are demonstrated through a number of numerical examples.

METHODOLOGY

1 Winkler Model

Winkler [18] introduced a very condensed model, which in the literature is now called the Winkler foundation. Thanks to its simplicity, the Winkler model was used by practicing engineers for everyday construction. This model is created on the premise that the reaction forces per unit area are proportional to the deflection of the foundation itself at each point of the foundation. The characteristics of vertical deformation of the foundation are characterized by equal, independent, closely spaced, discrete and linearly elastic springs known as the sub-grade reaction modulus, K.

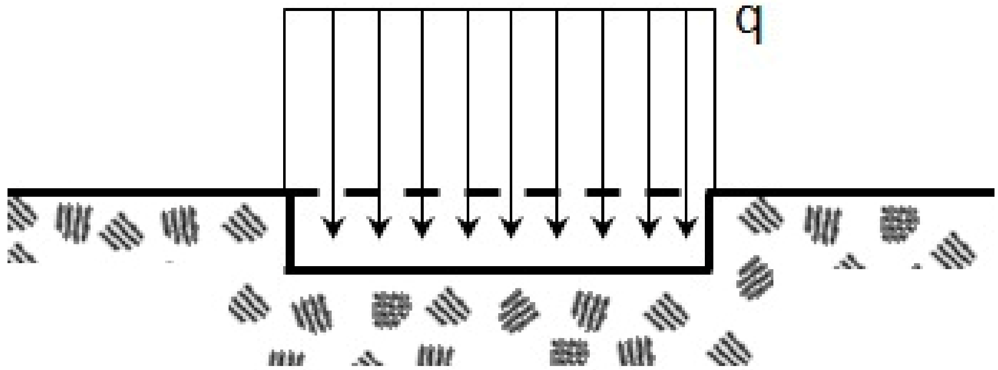

Fig. 1 Winkler foundation

The relationship between the exterior pressure and the deflection of the foundation surface under the pressure in the Winkler model is expressed as $q = Kw$ where w, the vertical displacement of the soil, at that point, is proportional to the contact pressure, q. The constant of proportionality, K is considered the "sub-grade reaction modulus" of the soil (Figure 1).

The foundation parameter modulus of sub-grade reaction can be evaluated from typical plate load tests. The modulus of sub-grade reaction (Ks) thus obtained is valid for 0.3 m wide footing. Corrections need to be applied for larger size footings used for structures. A simple correction for a ($B\ m \times B\ m$) square footing in a homogeneous cohesive soil with uniform elastic modulus, the modulus of sub-grade reaction can be estimated Terzaghi [19] as $K = Ks\ (0.3/B)$ and in cohesionless soils, $K = Ks\ [(B + 0.3)/2B]^2$ and for rectangular foundation of $B \times L$, $K = K_{B\times B}(1+B/L)/1.5$ where $K_{B\times B}$ is the modulus of sub-grade reaction for square footing with dimensions $B \times B$. According to Bowles [20], if q is the bearing capacity causing a settlement of 25 mm then, $K = 40\ qF$ where q is the allowable bearing pressure and F is the factor of safety.

2 Development of Governing Equation

The total potential energy of the soil-Beam system subjected to an uniformly distributed load (UDL) of q is given by $\Pi = U_{Beam}+U_{soil}-qw$

The strain energy in a linear spring is $\frac{1}{2}Kw^2$.

$$U_{soil} = \frac{1}{2}\int Kw^2\,dA = \frac{Kb}{2}\int w^2\,dx = \frac{k}{2}\int_0^L w^2 dx\ ;$$

$k = Kb;$ K – is the modulus of sub-grade reaction and b – is the beam width.

Consider an element of a thin beam, as shown in Figure 2, while bending. If w denotes the deflection of the beam along its length at any point x, $\partial w/\partial x$ denotes the slope of the deflected center-line. Due to the transverse displacement w, the axial displacement of a fiber centered at a distance z from the neutral axis u may be expressed as (point A moves to A') Rao [21], according to EBBT, a plane section of the beam remains plane after deformation.

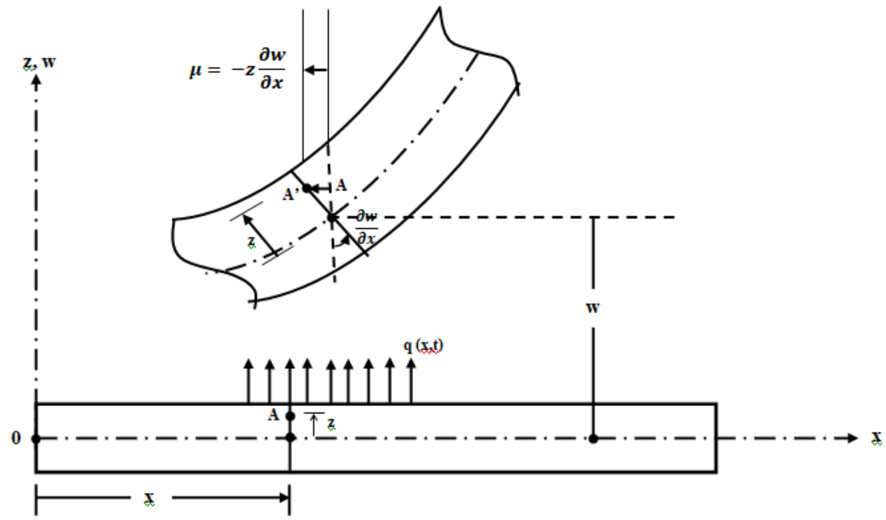

Fig. 2 Beam in bending

$$u = -z\frac{\partial w}{\partial x};\ \varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \therefore \sigma_x = E\varepsilon_x = -Ez\frac{\partial^2 w}{\partial x^2}$$

$\pi_0 = \frac{1}{2}\sigma_x\varepsilon_x$; Beam element energy density due to strain.

$$\pi_0 = \frac{1}{2}\left(-Ez\frac{\partial^2 w}{\partial x^2}\right)\left(-z\frac{\partial^2 w}{\partial x^2}\right) = \frac{1}{2}Ez^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2$$

Beam's total strain energy is given by,

$$\pi_b = \iiint_V \pi_0 dV = \int_0^L \frac{E}{2}\iint_A [z^2 dA]\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = \frac{1}{2}\int_0^L EI(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

where E - is the Young's modulus of beam material, I - is the moment of inertia of the beam.

$W = \int_0^L qwdx$; is the virtual work due to applied UDL, q, on the beam.

$$\therefore \Pi = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^L k\,w^2 dx$$

2.1 *Hamilton's Principle*

Lagrangian, L = T – U = Kinetic energy – Potential energy;

$$L = L\left(q_1, q_2, q_3 \ldots.\frac{dq_1}{dt}, \frac{dq_2}{dt}, \frac{dq_3}{dt}\right)\ and\ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\frac{dq_i}{dt}}\right) = 0;$$

is known as Euler-Lagrange equation and the governing differential equations of Lagrangian of motion.

Beam kinetic energy can be expressed as, $T = \frac{1}{2}\int_0^L m(x)\left(\frac{\partial w}{\partial t}\right)^2 dx$

In case of Euler beam on Winkler foundation $L = \frac{m}{2}\left(\frac{\partial w}{\partial t}\right)^2 - \frac{EI}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - \frac{kw^2}{2}$

In calculus of variations the order of integration with respect to t and x can be interchanged and the operator's δ and d/dx or δ and d/dt are commutative.

$$\therefore \delta\pi = \delta\int_{t_1}^{t_2}\int_0^L\left[\frac{EI}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{kw^2}{2}\right]dx\,dt = \frac{\partial}{\partial x}\int_{t_1}^{t_2}\int_0^L\left[\frac{EI}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{kw^2}{2}\right]dx\,dt$$

$$\therefore \delta\pi = \int_{t_1}^{t_2}\int_0^L\left[\frac{EI}{2}\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + kw\frac{\partial w}{\partial x}\right]dx\,dt = \int_{t_1}^{t_2}\int_0^L\frac{EI}{2}\left[2\frac{\partial^2 w}{\partial x^2}\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right) + kw\frac{\partial w}{\partial x}\right]dx\,dt$$

$$\therefore \delta\pi = \int_{t_1}^{t_2}\int_0^L \left[\left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right) + kw\frac{\partial w}{\partial x}\right] dx\,dt$$

$$\int_{t_1}^{t_2}\int_0^L \left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right) dx\,dt = \int_{t_1}^{t_2}\left[\left(EI\frac{\partial^2 w}{\partial x^2}\right)\int_0^L \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right) dx - \int_0^L \left\{\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\int_0^L \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\right) dx\right\} dx\right] dt$$

$$\therefore \delta\pi = \int_{t_1}^{t_2} EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2\Bigg|_0^L dt - \int_{t_1}^{t_2}\int_0^L \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 w}{\partial x^2} dxdt + \int_{t_1}^{t_2}\int_0^L kw\frac{\partial w}{\partial x} dx\,dt;$$

$$Now\ \int_{t_1}^{t_2}\int_0^L \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 w}{\partial x^2} dxdt = \int_{t_1}^{t_2}\left[\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\int_0^L \frac{\partial^2 w}{\partial x^2} dx - \int_0^L \left\{\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\int \frac{\partial^2 w}{\partial x^2} dx\right\} dx\right] dt$$

$$= \int_{t_1}^{t_2} \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial w}{\partial x}\Bigg|_0^L dt - \int_{t_1}^{t_2}\int_0^L \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial w}{\partial x} dxdt$$

$$\therefore \delta\pi = \int_{t_1}^{t_2} EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2\Bigg|_0^L dt - \int_{t_1}^{t_2} \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial w}{\partial x}\Bigg|_0^L dt + \int_{t_1}^{t_2}\int_0^L \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial w}{\partial x} dxdt + \int_{t_1}^{t_2}\int_0^L kw\frac{\partial w}{\partial x} dx\,dt$$

$$\therefore \delta\pi = \int_{t_1}^{t_2}\left[EI\frac{\partial^2 w}{\partial x^2}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L - \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w\Bigg|_0^L + \int_0^L \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w dx + \int_0^L kw\delta w dx\right] dt$$

$$\therefore \delta T = \delta\int_{t_1}^{t_2}\int_0^L \frac{m(x)}{2}\left(\frac{\partial w}{\partial t}\right)^2 dxdt = \int_{t_1}^{t_2}\int_0^L \frac{m}{2}2\frac{\partial w}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right) dxdt$$

$$\therefore \delta T = \int_0^L \left[m\frac{\partial w}{\partial t}\int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right) dt - \int_{t_1}^{t_2}\left\{m\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right)\int \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right) dt\right\} dt\right] dx;$$

$$\therefore \delta T = \int_0^L \left[m\frac{\partial w}{\partial t}\delta w\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\frac{\partial^2 w}{\partial t^2}\delta w dt\right] dx = -\int_0^L\int_{t_1}^{t_2} m\frac{\partial^2 w}{\partial t^2}\delta w dtdx\ as\ \delta w is\ zero\ at\ t = t_1\ and\ t = t_2$$

$$\therefore \delta W = \delta\int_{t_1}^{t_2}\int_0^L qwdxdt = \int_{t_1}^{t_2}\int_0^L q\delta wdxdt$$

Using the Hamilton's principle, one gets, $\delta L = \int_{t_1}^{t_2}(\delta T - \delta\pi + \delta W)dt$

$$\therefore \delta L = \int_{t_1}^{t_2}\left[-\int_0^L m\frac{\partial^2 w}{\partial t^2}\delta w dx - \left\{EI\frac{\partial^2 w}{\partial x^2}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L - \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w\Bigg|_0^L + \int_0^L \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w dx\right\} - \int_0^L kw\delta w dx + \int_0^L q\delta w dx\right] dt$$

$$\int_{t_1}^{t_2}\left[-\int_0^L m\frac{\partial^2 w}{\partial t^2}\delta w dx - \left\{EI\frac{\partial^2 w}{\partial x^2}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L - \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w\Bigg|_0^L + \int_0^L \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w dx\right\} - \int_0^L kw\delta w dx + \int_0^L q\delta w dx\right] dt = 0$$

$$\int_{t_1}^{t_2}\left[\int_0^L \left(-m\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) - kw + q\right)\delta w dx - \left\{EI\frac{\partial^2 w}{\partial x^2}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L - \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w\Bigg|_0^L\right\}\right] dt = 0$$

Since the time integration must hold for any arbitrary initial and final times, t_1 and t_2, respectively, we have that

$$\int_0^L \left(-m\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) - kw + q\right)\delta w dx - \left\{EI\frac{\partial^2 w}{\partial x^2}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\Bigg|_0^L - \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w\Bigg|_0^L\right\} = 0$$

Then, because the variations at $x = 0$ and $x = L$ are also arbitrary, the integration term and the boundary relations must be independently equal to zero.

$$\int_0^L \left(-m\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) - kw + q\right)\delta w dx = 0$$

Hence, we get the following three equations

$$m\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + kw - q = 0$$

$$m\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + c\frac{\partial w}{\partial t} + kw - q = 0$$

where '*c*' is the damping co-efficient.

For free vibration analysis $q = 0$ and the equation of motion for beam on elastic foundation is

$$m\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)+c\frac{\partial w}{\partial t}+kw=0 \quad (1)$$

$$EI\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^L=0 \quad (2)$$

$$and\ \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\delta w\Big|_0^L=0 \quad (3)$$

Equation (1) is the equation of motion for the beam's transverse vibration, and Equations (2) and (3) reflects the conditions of the boundaries. It can be seen that Equation (2) requires that either $EI\frac{\partial^2 w}{\partial x^2}=0\ or\ \delta\left(\frac{\partial w}{\partial x}\right)=0\ at\ x=0\ and\ x=L$. Equation (3) requires that either $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)=0\ or\ \delta w=0\ at\ x=0\ and\ x=L$

Common boundary conditions can be satisfied by the Equations (2) and (3):

2.2. *Boundary Conditions*

1. Fixed or clamped (C) end $w=0\ and\ \frac{\partial w}{\partial x}=0$.
2. Pinned or hinged (S) end $w=0\ and\ \frac{\partial^2 w}{\partial x^2}=0$.
3. Free (F) end $\frac{\partial^2 w}{\partial x^2}=0\ and\ \frac{\partial^3 w}{\partial x^3}=0$.

3 Characteristic Equation

Using the Hamilton principle Petyt [22], the equations of motion of a beam-soil system subjected to free vibration without damping is express as

$$[M]\{\ddot{w}\}+[K_{bs}]\{w\}=\{0\} \quad (4)$$

where $[K_{bs}]$ and $[M]$ is the stiffness matrix and mass matrix of the beam-soil system, w and $\ddot{w}$ is the displacement and acceleration of beam respectively. Obtain the natural frequencies and the modes of vibration by solving the generalized Eigen value problem Hinton [23].

For free vibration analysis assuming a harmonic motion

$$\{w\}=\{\overline{w}\}e^{i\omega t}\ and\ \{\ddot{w}\}=-\omega^2\{\overline{w}\}e^{i\omega t}$$

where $\{\overline{w}\}$ is the amplitude of $\{w\}$.

After substitution $\{w\}$ and $\{\ddot{w}\}$ in Equation (4) get

$$([K_{bs}]-\omega^2[M])\{\overline{w}\}=\{0\}$$

Non – trivial solution of Equation (5) requires that

$$|([K_{bs}]-\omega^2[M])|=0 \quad (5)$$

the characteristic equation. Where 'ω' is the natural frequency.

The mode shapes and natural frequency from the Eigen value solution of Equation (5) for the beam soil system can be obtained.

4 Finite Element Formulation

The finite element method was used to solve the problem. The derivations of the beam and foundation stiffness, as well as the force vector, are all included. In conjunction with the Winkler foundation model, the strain energy and virtual work methods are used. To make matrix inversion and solution of the derived equations easier, a computer program written in Matlab is created. Consider the three nodded beam with nodal displacements and forces as shown in Figure 3. This section has nodes at $x=(-L/2,\ 0,\ L/2)$. For more details of two DOF per node three nodded beam is available in [1, 13, 14, 24].

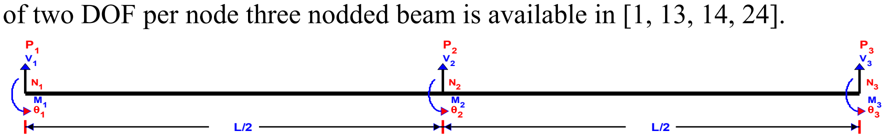

Fig. 3 Three-nodded beam element components for displacement and force

The element has six DOF in total, defined by the vector of nodal displacement.

$$q=[v_1\quad \theta_1\quad v_2\quad \theta_2\quad v_3\quad \theta_3]^T$$

The corresponding nodal force vector $P = [P_1 \quad M_1 \quad P_2 \quad M_2 \quad P_3 \quad M_3]^T$

Using the boundary conditions of Figure 3, the nodal displacement can be expressed in terms of the nodal displacement vector q in the form $v = Nq$, where N is the shape function and q is the nodal displacement.

Three nodded two DOF per node beam components have shape functions

$$N_1 = \frac{4x^2}{L^2} - \frac{10x^3}{L^3} - \frac{8x^4}{L^4} + \frac{24x^5}{L^5}; N_2 = \frac{x^2}{2L} - \frac{x^3}{L^2} - \frac{2x^4}{L^3} + \frac{4x^5}{L^4}$$

$$N_3 = \frac{16x^4}{L^4} - \frac{8x^2}{L^2} + 1; N_4 = x - \frac{8x^3}{L^2} + \frac{16x^5}{L^4}$$

$$N_5 = \frac{4x^2}{L^2} + \frac{10x^3}{L^3} - \frac{8x^4}{L^4} - \frac{24x^5}{L^5}; N_6 = \frac{2x^4}{L^3} - \frac{x^3}{L^2} - \frac{x^2}{2L} + \frac{4x^5}{L^4}$$

Strain energy of beam element , $U_b = \int_{-\frac{L}{2}}^{\frac{L}{2}} EI\{d\}^T[N'']^T[N'']\{d\}\,dx = \frac{1}{2}\{d\}^T[K_b]\{d\}$

where beam element stiffness is $K_b = EI\int_{-\frac{L}{2}}^{\frac{L}{2}}[N'']^T[N'']\,dx$

The flexural stiffness matrix for the three nodded beam elements is

$$K_b = \frac{EI}{35L^3}\begin{bmatrix} 5092 & 1138L & -3584 & 1920L & -1508 & 242L \\ 1138L & 332L^2 & -896L & 320L^2 & -242L & 38L^2 \\ -3584 & -896L & 7168 & 0 & -3584 & 896L \\ 1920L & 320L^2 & 0 & 1280L^2 & -1920L & 320L^2 \\ -1508 & -242L & -3584 & -1920L & 5092 & -1138L \\ 242L & 38L^2 & 896L & 320L^2 & -1138L & 332L^2 \end{bmatrix}$$

Strain energy of linear spring $U_r = \frac{1}{2}\int K\{d\}^T[N]^T[N]\{d\}\,dA = \frac{1}{2}\{d\}^T[K_f]\{d\}$

In which the foundation stiffness matrix for the element is,

$$[K_f] = \int_{-\frac{L}{2}}^{\frac{L}{2}} K[N]^T[N]\,dA = Kb\int_{-\frac{L}{2}}^{\frac{L}{2}}[N]^T[N]\,dx = k\int_{-\frac{L}{2}}^{\frac{L}{2}}[N]^T[N]\,dx$$

where $[N]$ is the shape function matrix of the beam element.

The stiffness matrix of foundation for the three nodded beam elements is

$$K_f = \frac{kL}{13860}\begin{bmatrix} 2092 & 114L & 880 & -160L & 262 & -29L \\ 114L & 8L^2 & 88L & -12L^2 & 29L & -3L^2 \\ 880 & 88L & 5632 & 0 & 880 & -88L \\ -160L & -12L^2 & 0 & 128L^2 & 160L & -12L^2 \\ 262 & 29L & 880 & 160L & 2092 & -114L \\ -29L & -3L^2 & -88L & -12L^2 & -114L & 8L^2 \end{bmatrix}$$

Hence the element stiffness matrix is

$$\therefore [k_e] = [K_b] + [K_f]$$

By using standard finite element procedure, the global stiffness matrix is obtained

$$\therefore [K_{bs}] = \sum_{1}^{n}([K_b] + [K_f])$$

$$\text{Kinetic energy, } T = \frac{1}{2}\rho A\int_{-\frac{L}{2}}^{\frac{L}{2}}\left(\frac{\partial w}{\partial t}\right)^2 dx$$

$$T = \frac{1}{2}\{d'\}^T\rho A\int_{-\frac{L}{2}}^{\frac{L}{2}}[N]^T[N]dx\{d'\}$$

For element mass matrix

$$m = \rho A \int_{-\frac{L}{2}}^{\frac{L}{2}} [N]^T \, [N] dx$$

The consistent mass matrix for the three nodded beam elements is

$$m = \frac{\rho A L}{13860} \begin{bmatrix} 2092 & 114L & 880 & -160L & 262 & -29L \\ 114L & 8L^2 & 88L & -12L^2 & 29L & -3L^2 \\ 880 & 88L & 5632 & 0 & 880 & -88L \\ -160L & -12L^2 & 0 & 128L^2 & 160L & -12L^2 \\ 262 & 29L & 880 & 160L & 2092 & -114L \\ -29L & -3L^2 & -88L & -12L^2 & -114L & 8L^2 \end{bmatrix}$$

By using standard finite element procedure, the global mass matrix is obtained

$$\therefore [M] = \sum_{1}^{n} m$$

Before solving the characteristic equation of system, one needs to be applied boundary conditions.

RESULTS AND DISCUSSION

1 Convergence Study and Validation of the Present Formulation

The following dimensionless parameters are defined for results comparison

- Natural frequency parameter, λ *or* $\beta = [\omega L^2 \sqrt{(\rho A/EI)}]^{1/2}$;
- Foundation parameter, $Kw = kL^4/EI$;

An example has been chosen from the study done by Kacar et al. [12] for convergence study as well as for validation of the present formulation. The dimensions of the simply supported beam for verification were taken as follows: L = 1 m, beam thickness, H = 0.05 m, material properties: Young's modulus, E = 20000 MPa, ρ = 2500 kg/m³. Kw = 10.

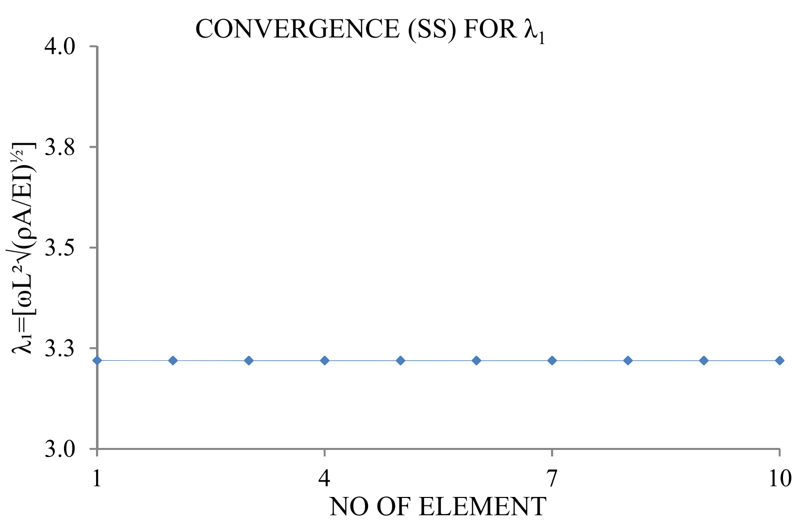

Fig. 4 Convergence study for natural frequency for 1st mode of a simply supported (S - S) beam

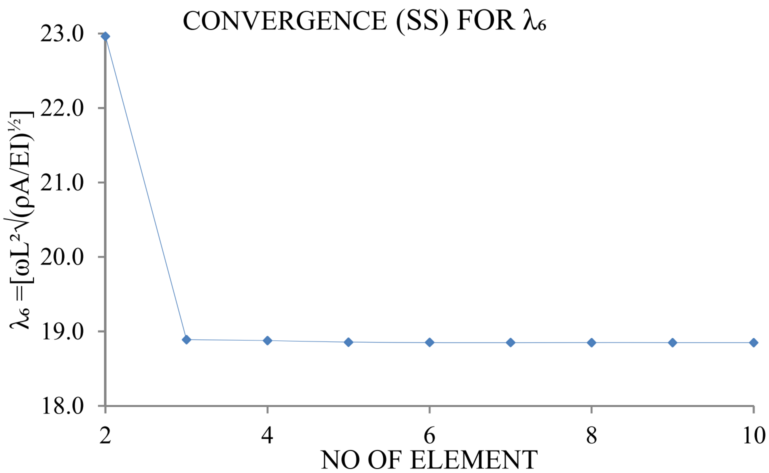

Fig. 5 Convergence study for natural frequency for sixth mode of a S - S beam

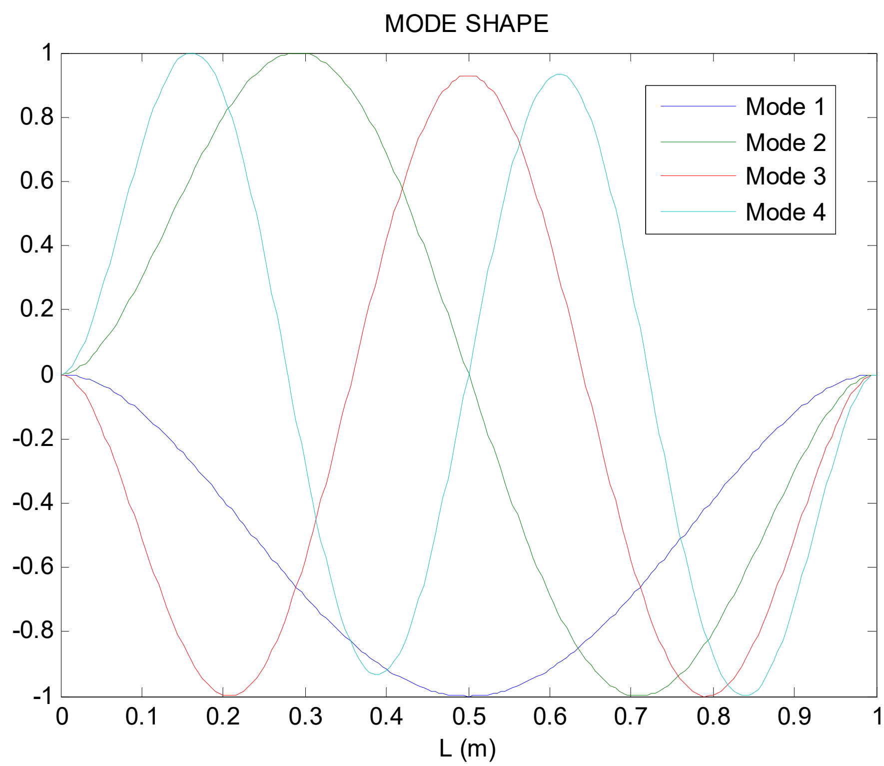

Fig. 6 Mode shape of clamped-clamped (C - C) beam

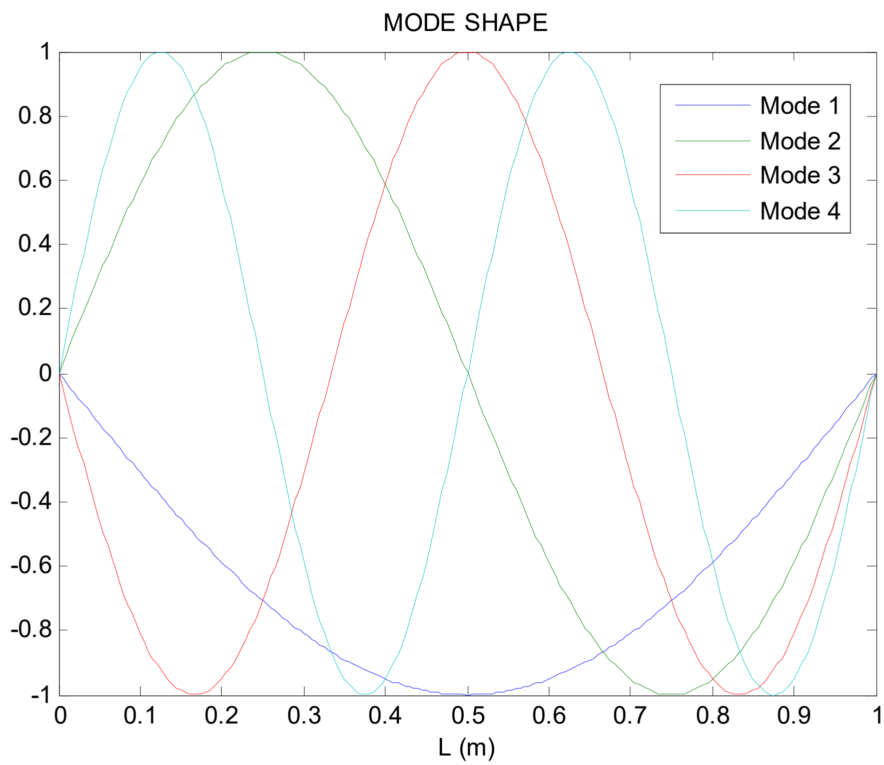

Fig. 7 Mode shape of S - S beam

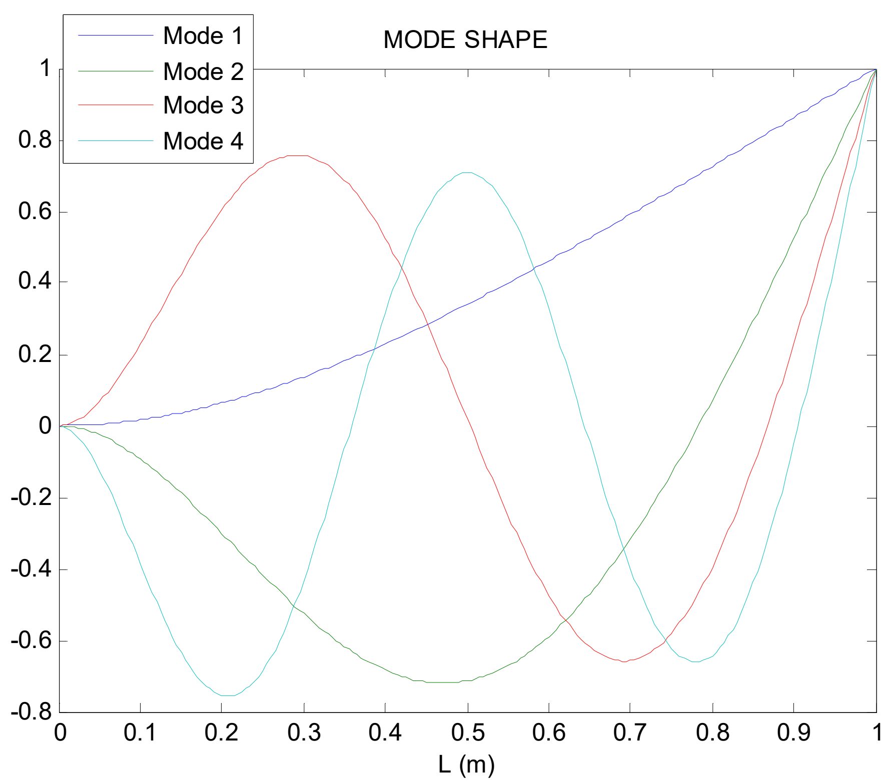

Fig. 8 Mode shape of clamped-free (C - F) beam

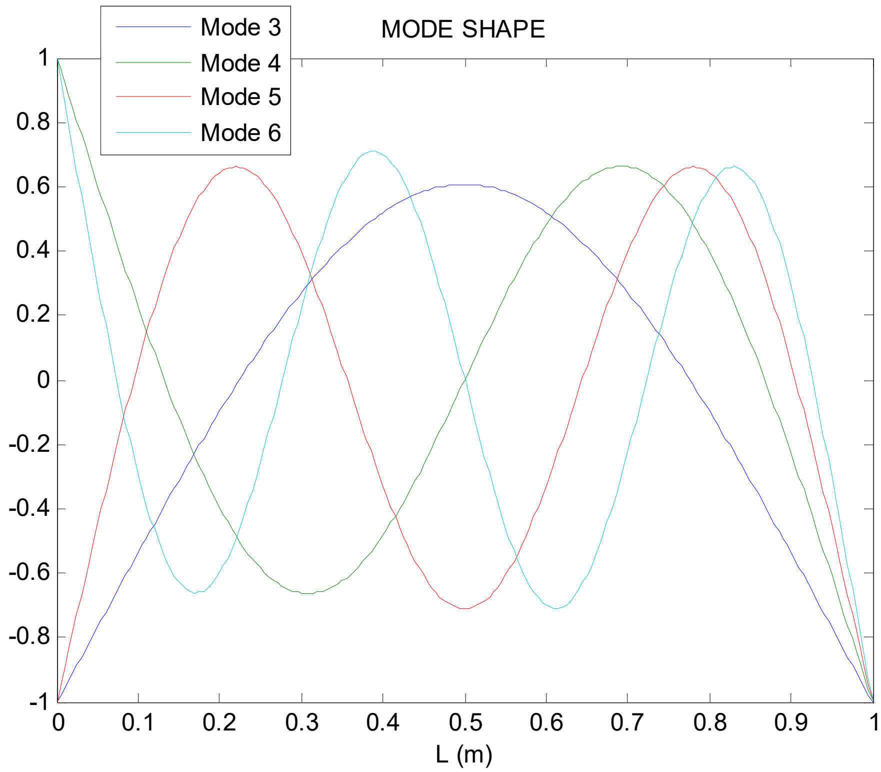

Fig. 9 Mode shape of free-free (F - F) beam

Convergence study for modes 1 and 6 shown in Figures 4 and 5 and also the mode shape for various boundary conditions are given in Figures 6-9. The obtained frequency parameter listed in Table 1 and has been compared with Kacar et al. [12] show excellent agreement. It is also observed that only one element or three nodes is sufficient for reasonable result for the first mode and up to the sixth mode achieved only five elements or eleven nodes is sufficient.

In case of F-F beam there will be two rigid body modes in a normal modes analysis with no loads or constraints: one translation and other rotation modes. The modal frequency of the first two modes will be zero or close to zero. This refers to a rigid body motion rather than a vibrating system. In the present study for F-F beam first significant mode is mode 3.

Table 1: Frequency parameter for validation of the present formulation

THEORY	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Present study (PS)	3.2193	6.2932	9.4278	12.5678	15.7109	18.8555	22.0114	25.1909
Kacar et al. [12]	3.2193	6.2932	9.4278	12.5676	15.7086	18.8499	21.9914	25.1329
Analytical	3.2193	6.2932	9.4278	12.5676	15.7086	18.8499	21.9914	25.1329
Difference (%)	0	0	0	0.0016	0.0149	0.0298	0.0907	0.2308

This section starts with some comparisons with similar studies done by other researchers are made. The present method can be applied to analyze the free vibration analysis of a beam with various boundary conditions, including free-free.

An example has been chosen from the study done by Friswell et al. [4] beam on Winkler foundation. The foundation, beam's material properties and load are presented in Table 2. The obtained result listed in Table 3 and has been compared with analytical and as well as Friswell et al. [4] and result show full agreement with analytical result.

Table 2: The foundation property and beam's properties

Beam length	6.096 m.
Beam moment of inertia	0.001439 m^4
Elastic modulus of the beam	24.82×10^6 KN/m^2
Mass per meter of the beam	446.3 kg/m
Foundation modulus, k = Kb	16550 KN/m^2

Table 3: Comparisons of the natural frequencies (Hz) of vibration for the Simple supported beam

Mode No	Analytical	Friswell et al. [4]	Present Study
1	32.898	32.898	32.898
2	56.808	56.812	56.808
3	111.90	111.95	111.899
4	193.76	194.08	193.768

Table 1 and Table 3 show how superior the present formulation is.

Quintic displacement functions, as shown in the Tables 1 and 3 above, provide satisfactory precision. The method is also accurate for the free vibrational modes of a beam resting on an elastic foundation, which is the study's major focus and where the results are presented in Tables 1 and 3 alongside comparisons to the available literature.

2 Parametric Study

For parametric study the data has been taken as E = 20G Pa and Kw = 20, 40, 50 and various boundary condition (B.C.) and L/d = 100.

The dimensions of the beam for parametric study were taken as follows: L= 5 m, thickness, d = 50 mm, and material properties: Young's modulus, E = 20 GPa and density, ρ = 2500 kg/m^3 and the non-dimensional value of Kw = 20, 40, 80. The actual values of the Winkler coefficients for Kw = 20, were k = 6666.67 N/m^2, and various boundary condition and L/d = 100. Results for parametric study are presented in Table 4 and Figures 10 and 11. In case of F-F beam frequency parameter is mode 3 to mode 12.

Table 4: Parametric study E = 20 GPa and Kw = 20, 40 and 80

B.C.	Kw	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
C-C	20	4.7766	7.8635	10.9994	14.1390	17.2836	20.4324	23.5979	26.8045	30.1584	33.1830
	40	4.8218	7.8738	11.0032	14.1408	17.2846	20.4330	23.5982	26.8047	30.1585	33.1831
	80	4.9087	7.8942	11.0107	14.1444	17.2865	20.4341	23.5990	26.8052	30.1589	33.1834
S-S	20	3.2917	6.3032	9.4308	12.5691	15.7116	18.8559	22.0116	25.1911	28.4169	31.4825
	40	3.4238	6.3231	9.4367	12.5716	15.7129	18.8566	22.0121	25.1914	28.4171	31.4827
	80	3.6496	6.3623	9.4486	12.5766	15.7154	18.8581	22.0130	25.1920	28.4175	31.4830
C-F	20	2.3851	4.7417	7.8651	10.9993	14.1391	17.2841	20.4344	23.6037	26.8173	30.1832
	40	2.6900	4.7879	7.8753	11.0031	14.1409	17.2851	20.4350	23.6041	26.8176	30.1834
	80	3.1001	4.8765	7.8957	11.0106	14.1444	17.2870	20.4362	23.6049	26.8181	30.1837
F-F	20	2.1147	2.1147	4.7766	7.8635	10.9994	14.1392	17.2846	20.4365	23.6096	26.8303
	40	2.5149	2.5149	4.8218	7.8738	11.0032	14.1410	17.2856	20.4371	23.6100	26.8305
	80	2.9907	2.9907	4.9087	7.8942	11.0107	14.1445	17.2875	20.4383	23.6107	26.8311
C-S	20	4.0067	7.0827	10.2149	13.3541	16.4972	19.6438	22.8037	25.9943	29.2574	32.3813
	40	4.0823	7.0967	10.2196	13.3562	16.4983	19.6445	22.8042	25.9946	29.2576	32.3814
	80	4.2219	7.1245	10.2289	13.3604	16.5006	19.6458	22.8050	25.9952	29.2580	32.3817
S-F	20	2.1147	4.0067	7.0827	10.2149	13.3541	16.4976	19.6453	22.8084	26.0051	29.2775
	40	2.5149	4.0823	7.0967	10.2196	13.3562	16.4987	19.6460	22.8088	26.0054	29.2777
	80	2.9907	4.2219	7.1245	10.2289	13.3604	16.5009	19.6473	22.8096	26.0059	29.2781

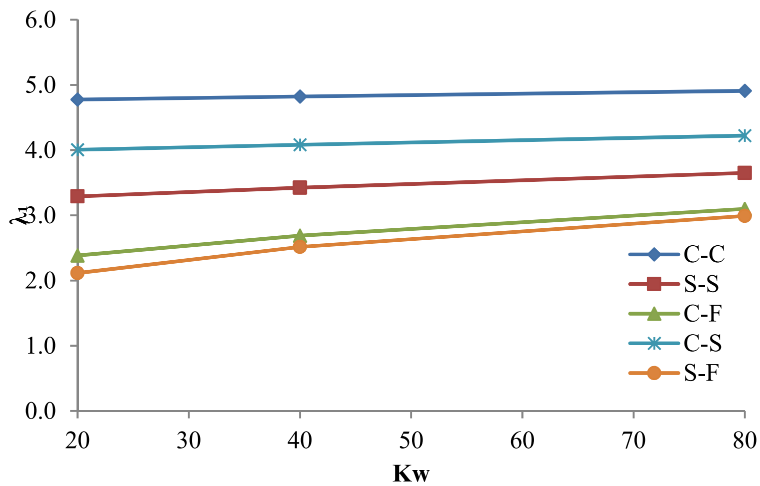

Fig. 10 First mode frequency parameter vs foundation parameter

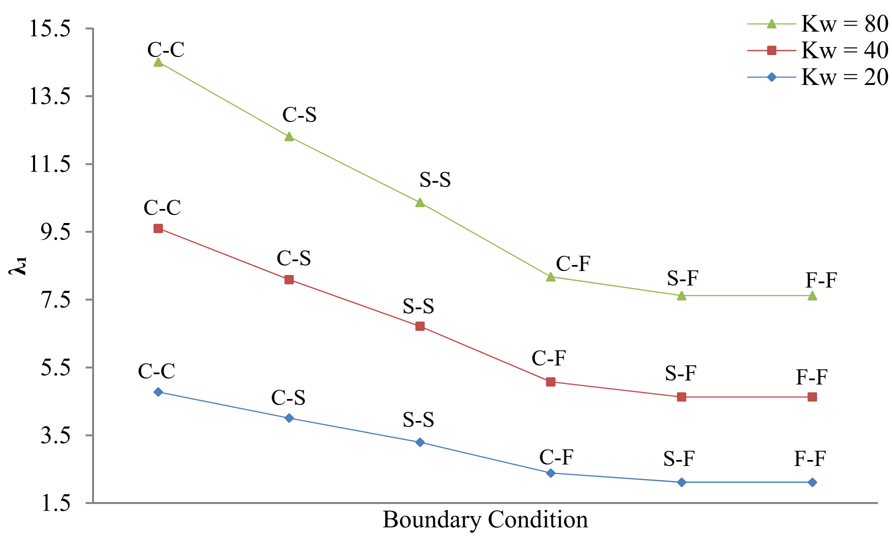

Fig. 11 Effect of boundary condition on frequency parameter

Table 4 show the variation of frequency parameter with modulus of sub-grade reaction. It is observed that the frequency parameter increases with increase of sub-grade reaction, as the modulus of sub-grade reaction increases, the foundation stiffness as well as the beam's overall stiffness increases, resulting in a greater frequency response. It is seen that the increase in frequency parameter is 1-2% when the foundation parameter increase cent percent.

Figures 4, 5 shows the rapid convergence rate of the present formulation.

It is observed from Table 4 and Figures 10, 11 that with increase of constrains on the edge the frequency parameter increases because of beam flexural rigidity increases as edge constraints increases and hence higher frequency response.

CONCLUSIONS

The dynamic response of an Euler-Bernoulli beam under free vibration is the subject of this research. Winkler elastic foundation supports the beam with constant cross-sectional area over its length. Winkler modelling is commonly used to represent beams and pipes that are supported by an elastic soil. Such modeling introduces the elastic foundation by a set of mutually independent spring elements. In the present work, the elastic coefficient of the spring set is constant throughout the major axis of the beam. In the present study, the free vibration analysis of beam resting on Winkler foundation is investigated using quintic displacement functions, the result obtained has excellent agreement with the exact solution as well as the referred result and very high convergence rate regardless of boundary conditions, length to depth ratio and foundation parameter. It is seen that the three nodded two DOF per node C^1 compatible type beam element is simple and can be used effectively and easily for free vibration analysis of beams on Winkler foundation under any type of boundary conditions. The B. C's influence is more pronounced for a higher mode number than the lower mode. In the light of the above results, it may be concluded that the present formulation has generates free vibration responses of beams on Winkler foundation with few elements and within few seconds and saves computer memory, resources and time.

Future Scope of the Work.

- The present research work can be extended to consider the effect of layered soil or Gibson soil.
- The analysis can be further extended to account for time varying elastic properties of the soil.
- The model can be extended to consider forced vibration and stability analysis of beams on elastic foundation.
- The model can be extended to two-parameter and continuum-based foundation model.
- The model can be extended to consider analysis of anisotropic, laminated and functionally graded beams on elastic foundation.

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