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ON SINGLE STEP ALGORITHMS IN THE SOLUTION OF DYNAMIC LINEAR AND NONLINEAR PROBLEMS

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Abstract

Dynamic analysis of structures for the evaluation of the timewise response is often conducted using direct integration of equations of motion. A number of methods have been proposed for direct integration of these equations of motion. In this study two general families of single step algorithms for direct integration are considered *viz* the Generalized Newmark (GN) algorithm emanating from the generalization of the Newmark method and Single Step (SS) algorithm due to Zienkiewicz, Wood and their coworkers. From these generalized forms a number of new and existing algorithms can be simulated. Stepwise procedure for GN and SS family of algorithms are developed in predictor – corrector form. From these procedures Newmark's unconditionally stable algorithm, Houbolt (originally multistep) algorithm and Wilson θ method are simulated. These simulated procedures are applied to linear as well as in nonlinear problems. Each family is seen to have advantages as well as disadvantages over the other. It is seen that elastoplasticity does not alter the predominant frequency response of the system. It, however, introduces a zero frequency and some low frequency components in the response. It is also seen that strain softening plasticity can be successfully used with dynamic problems.

INTRODUCTION

The equilibrium equations of a multidegree freedom system in motion are of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, $\mathbf{f}$ is the time varying external load vector and $\mathbf{x}$, $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ are the displacement, velocity and acceleration vectors of the nodes in the discretised element assemblage. In direct integration the equations (1) are integrated using a numerical step by step procedure. The term direct means that prior to numerical integration, no transformation of the equations into a different form is carried out. In these methods the aim is to satisfy the equations (1)

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at discrete time steps at interval Δt. The methods in which the recurrence algorithms are valid within a single time step and relate the values $\mathbf{x}_{n+1}$, $\dot{\mathbf{x}}_{n+1}$, $\ddot{\mathbf{x}}_{n+1}$ at step $n+1$ to $\mathbf{x}_n$, $\dot{\mathbf{x}}_n$, $\ddot{\mathbf{x}}_n$ at step n are called *Single Step Methods*. In *Multistep Methods* $\mathbf{x}_{n+1}$ are related to $\mathbf{x}_n$, $\mathbf{x}_{n-1}$, $\mathbf{x}_{n-2}$ *etc.* without introducing explicitly the derivatives and assuming that each set is separated by an equal interval Δt. Such algorithms are, in general, less convenient to use than the single step procedures, as they do not permit an easy change of the time step magnitude. Also these methods require a greater degree of book-keeping as displacements of previous steps are required to be stored. Various such algorithms have been derived using the weighted residual process (Zienkiewicz and Wood, 1986) and it has been shown that these procedures have identical stability and accuracy properties as single step methods.

Zienkiewicz *et al.* (1984) used the term single step for the set of algorithms devised by them. In this study a broader meaning to this term is implied. All algorithms that relate values at step $n+1$ to values at step n have been termed as Single Step Algorithms and two generalized algorithms *viz.* (a) those due to Zienkiewicz *et al.* (1984) termed as SSpj and (b) those emanating from the generalization of Newmark method (Zienkiewicz and Katona, 1985) GNpj are studied. In GNpj and SSpj, p stands for the order of the polynomial of approximation of the variable $\mathbf{x}$ and j stands for the order of the differential equations. Since this study is confined to vibrational problems, the value for j equals 2.

GENERALIZED NEWMARK (GN) PROCEDURE

This procedure applies Taylor series approach to derive a general form of single step algorithms that can be considered to be a generalization of Newmark method (Zienkiewicz and Wood, 1986). It results in a scheme which is not self starting. In the derivation one considers the satisfaction of the governing equations (1) only at the end points of the interval Δt and writes (Zienkiewicz and Katona, 1985)

$$\mathbf{M}\ddot{\mathbf{x}}_{n+1} + \mathbf{C}\dot{\mathbf{x}}_{n+1} + \mathbf{K}\mathbf{x}_{n+1} = \mathbf{f}_{n+1} \tag{2}$$

with appropriate approximations for the values of $\mathbf{x}_{n+1}$, $\dot{\mathbf{x}}_{n+1}$ and $\ddot{\mathbf{x}}_{n+1}$. If one considers the Taylor series expansion the derivatives can be written as

$$\begin{aligned}
\mathbf{x}_{n+1} &= \mathbf{x}_n + \Delta t\dot{\mathbf{x}}_n + \cdots + \frac{\Delta t^p}{p!}\overset{p}{\mathbf{x}}_n + \beta_p\frac{\Delta t^p}{p!}(\overset{p}{\mathbf{x}}_{n+1} - \overset{p}{\mathbf{x}}_n) \\
&= \tilde{\mathbf{x}}_{n+1} + \beta_p\frac{\Delta t^p}{p!}\overset{p}{\mathbf{x}}_{n+1} \\
\dot{\mathbf{x}}_{n+1} &= \dot{\mathbf{x}}_n + \Delta t\ddot{\mathbf{x}}_n + \cdots + \frac{\Delta t^{p-1}}{(p-1)!}\overset{p}{\mathbf{x}}_n + \beta_{p-1}\frac{\Delta t^{p-1}}{(p-1)!}(\overset{p}{\mathbf{x}}_{n+1} - \overset{p}{\mathbf{x}}_n) \\
&= \dot{\tilde{\mathbf{x}}}_{n+1} + \beta_{p-1}\frac{\Delta t^{p-1}}{(p-1)!}\overset{p}{\mathbf{x}}_{n+1} \\
&\vdots \quad\quad \vdots \\
\overset{p-1}{\mathbf{x}}_{n+1} &= \overset{p-1}{\mathbf{x}}_n + \Delta t\overset{p}{\mathbf{x}}_n + \beta_1\Delta t(\overset{p}{\mathbf{x}}_{n+1} - \overset{p}{\mathbf{x}}_n)
\end{aligned}$$

$$= {}^{p-1}\tilde{\mathbf{x}}_{n+1} + \beta_1 \Delta t^p \overset{p}{\tilde{\mathbf{x}}}_{n+1} \tag{3}$$

where

$$\overset{p}{\mathbf{x}}_n = \frac{d^p \mathbf{x}_n}{dt^p}$$

In equation (3) for a polynomial of degree p, a Taylor series remainder term has effectively been allowed in each of the expansions for the functions and its derivatives with parameter $\beta_j, j = 1, 2, \cdots, p$ which can be chosen to give good approximation properties to the algorithm.

Insertion of the first three equations of (3) into equation (2) gives a single equation from which $\overset{p}{\tilde{\mathbf{x}}}_{n+1}$ can be found. When this is determined $\mathbf{x}_{n+1}$ to ${}^{p-1}\tilde{\mathbf{x}}_{n+1}$ can be evaluated using equations (3). The expression is

$$\begin{aligned} \overset{p}{\tilde{\mathbf{x}}}_{n+1} &= \left(\mathbf{M}\Delta t^{p-2} \frac{\beta_{p-2}}{(p-2)!} + \mathbf{C}\Delta t^{p-1} \frac{\beta_{p-1}}{(p-1)!} + \mathbf{K}\Delta t^p \frac{\beta_p}{p!} \right)^{-1} \\ &\quad \times \left(\mathbf{f}_{n+1} - \mathbf{M}\tilde{\ddot{\mathbf{x}}}_{n+1} - \mathbf{C}\dot{\tilde{\mathbf{x}}}_{n+1} - \mathbf{K}\tilde{\mathbf{x}}_{n+1} \right) \end{aligned} \tag{4}$$

It can be easily shown that the commonly used Newmark method can be derived from the above generalized procedure. The above algorithm applies to both implicit and explicit schemes. In terms of Generalized Newmark method an explicit scheme is simply defined by $\beta_p = 0$ for any order of p (Zienkiewicz and Katona, 1985). Conversly, an implicit scheme is defined by $\beta_p \neq 0$, irrespective of the remaining integration parameters.

Algorithm for GN22 and GN33

A detailed implementation algorithm (Barthwal, 1992) for GN22 and GN33 is now discussed. The algorithm is in the predictor-corrector form which is particularly advantageous in nonlinear analysis. The changes required for nonlinear analysis are discussed in subsequent section.

(1) Begin predictor phase
Set iteration counter $i = 0$
if $p = 2$ *then*
$\mathbf{x}^i_{n+1} = \mathbf{x}_n + \Delta t \dot{\mathbf{x}}_n + (1 - \beta_2)(\Delta t^2/2)\ddot{\mathbf{x}}_n$
$\dot{\mathbf{x}}^i_{n+1} = \dot{\mathbf{x}}_n + (1 - \beta_1)\Delta t \ddot{\mathbf{x}}_n$
$\tilde{\mathbf{x}}_{n+1} = \mathbf{x}^i_{n+1}$
$\dot{\tilde{\mathbf{x}}}_{n+1} = \dot{\mathbf{x}}^i_{n+1}$
$\tilde{\ddot{\mathbf{x}}}^i_{n+1} = (\mathbf{x}^i_{n+1} - \tilde{\mathbf{x}}_{n+1})(2/(\beta_2 \Delta t^2)) = 0$
else
$\mathbf{x}^i_{n+1} = \mathbf{x}_n + \Delta t \dot{\mathbf{x}}_n + (\Delta t^2/2)\ddot{\mathbf{x}}_n + (1 - \beta_3)(\Delta t^3/6)\overset{3}{\mathbf{x}}_n$
$\dot{\mathbf{x}}^i_{n+1} = \dot{\mathbf{x}}_n + \Delta t \ddot{\mathbf{x}}_n + (1 - \beta_2)(\Delta t^2/2)\overset{3}{\mathbf{x}}_n$
$\tilde{\mathbf{x}}_{n+1} = \mathbf{x}^i_{n+1}$
$\dot{\mathbf{x}}_{n+1} = \dot{\mathbf{x}}^i_{n+1}$

$\ddot{\tilde{\mathbf{x}}}_{n+1} = \ddot{\mathbf{x}}^i_{n+1}$
$\overset{3}{\mathbf{x}}{}^i_{n+1} = (\mathbf{x}^i_{n+1} - \tilde{\mathbf{x}}_{n+1})(6/(\beta_3 \Delta t^3))$
endif

(2) Form effective stiffness matrix $\mathbf{K}^*$
if $p = 2$ *then*
$\mathbf{K}^* = \mathbf{M}(2/(\beta_2 \Delta t^2)) + \mathbf{C}(2\beta_1/(\Delta t \beta_2)) + \mathbf{K}$
else
$\mathbf{K}^* = \mathbf{M}(6\beta_1/(\Delta t^2 \beta_3)) + \mathbf{C}(3\beta_2/(\Delta t \beta_3)) + \mathbf{K}$
endif

(3) Evaluate residual forces
$\boldsymbol{\psi}_i = \mathbf{f}_{n+1} - \mathbf{M}\ddot{\mathbf{x}}^i_{n+1} - \mathbf{C}\dot{\mathbf{x}}^i_{n+1} - \mathbf{K}\mathbf{x}^i_{n+1}$

(4) Solve for incremental displacements $\Delta \mathbf{x}^i$
$\mathbf{K}^* \Delta \mathbf{x}^i = \boldsymbol{\psi}_i$

(5) Begin the corrector phase
$\mathbf{x}^{i+1}_{n+1} = \mathbf{x}^i_{n+1} + \Delta \mathbf{x}^i$
if $(p = 2)$ *then*
$\ddot{\mathbf{x}}^{i+1}_{n+1} = (\mathbf{x}^{i+1}_{n+1} - \tilde{\mathbf{x}}_{n+1})(2/(\beta_2 \Delta t^2))$
$\dot{\mathbf{x}}^{i+1}_{n+1} = \dot{\tilde{\mathbf{x}}}_{n+1} + \beta_1 \Delta t \ddot{\mathbf{x}}^{i+1}_{n+1}$
else
$\overset{3}{\mathbf{x}}{}^{i+1}_{n+1} = (\mathbf{x}^{i+1}_{n+1} - \tilde{\mathbf{x}}_{n+1})(6/(\beta_3 \Delta t^3))$
$\dot{\mathbf{x}}^{i+1}_{n+1} = \dot{\tilde{\mathbf{x}}}_{n+1} + \beta_2 (\Delta t^2/2) \overset{3}{\mathbf{x}}{}^{i+1}_{n+1}$
$\ddot{\mathbf{x}}^{i+1}_{n+1} = (\ddot{\tilde{\mathbf{x}}}_{n+1} + \beta_1 \Delta t \overset{3}{\mathbf{x}}{}^{i+1}_{n+1}$
endif

(6) *if* ($\Delta \mathbf{x}^i$ and/or $\boldsymbol{\psi}$ > specified tolerences) *then*
$i = i + 1$
go to (3)
else
go to (7)
endif

(7) *if* $(p = 2)$ *then*
$\mathbf{x}_{n+1} = \mathbf{x}^{i+1}_{n+1}$
$\dot{\mathbf{x}}_{n+1} = \dot{\mathbf{x}}^{i+1}_{n+1}$
else
$\mathbf{x}_{n+1} = \mathbf{x}^{i+1}_{n+1}$
$\dot{\mathbf{x}}_{n+1} = \dot{\mathbf{x}}^{i+1}_{n+1}$
$\ddot{\mathbf{x}}_{n+1} = \ddot{\mathbf{x}}^{i+1}_{n+1}$
endif

(8) Set $n = n + 1$ and begin next step

In the above algorithm the primary variable solved for is displacement rather than acceleration as would be the case if one were to use equation (4). This is advantageous for elastoplastic problems where the tangent stiffness matrix is displacement (or strain) dependent. However, the disadvantage is that the explicit scheme cannot be employed as this would require $\beta_2 = 0$ or $\beta_3 = 0$ for GN22 and GN32 respectively and would generate indeterminate values in steps 2 and 5 of the algorithm.

SS ALGORITHM

Zienkiewicz *et al.* (1984) gave a unified set of single step algorithms $SSpj$ using polynomials of degree p. They considered a single time interval Δt (Fig. 1) in which the

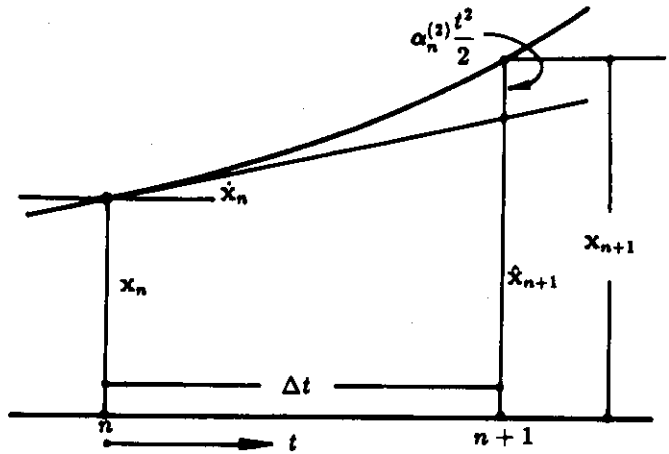


Figure 1: A second-order time step appproximation (Zienkiewicz *et al.* 1984).

variable $\mathbf{x}$ was approximated as a polynomial of degree p in time t taken as zero at n, i.e. $(0 \le t \le \Delta t)$ as

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_n + \dot{\mathbf{x}}_n t + \ddot{\mathbf{x}}_n \frac{t^2}{2!} + \cdots + \alpha_n^{(p)} \frac{t^p}{p!} \\ &= \sum_{q=0}^{p-1} \overset{q}{\mathbf{x}}_n \frac{t^q}{q!} + \alpha_n^{(p)} \frac{t^p}{p!} \end{aligned} \tag{5}$$

The unknown values $\mathbf{x}_{n+1}$, $\dot{\mathbf{x}}_{n+1}$ *etc.* are found using the known values at the start of the interval $\mathbf{x}_n$, $\dot{\mathbf{x}}_n$ *etc.*, and the as yet unknown vector $\alpha_n^{(p)}$ as

$$\begin{aligned} \mathbf{x}_{n+1} &= \sum_{q=0}^{p-1} \overset{q}{\mathbf{x}}_n \frac{\Delta t}{q!} + \alpha_n^{(p)} \frac{\Delta t^p}{p!} = \hat{\mathbf{x}}_{n+1} + \alpha_n^{(p)} \frac{\Delta t^p}{p!} \\ \dot{\mathbf{x}}_{n+1} &= \sum_{q=1}^{p-1} \overset{q}{\mathbf{x}}_n \frac{\Delta t^{q-1}}{q-1!} + \alpha_n^{(p)} \frac{\Delta t^{p-1}}{p-1!} = \hat{\dot{\mathbf{x}}}_{n+1} + \alpha_n^{(p)} \frac{\Delta t^{p-1}}{p-1!} \end{aligned} \tag{6}$$

To determine the unknown parameter $\alpha_n^{(p)}$, a weighted satisfaction of the equation of motion was assumed (Zienkiewicz *et al.*, 1984) *i.e.*

$$\int_0^{\Delta t} \mathbf{W}(\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} - \mathbf{f})\, dt = 0 \tag{7}$$

and a definition of following form employed

$$\frac{\int_0^{\Delta t} \mathbf{W} t^q\, dt}{\int_0^{\Delta t} \mathbf{W}\, dt} = \theta_q \Delta t^q \qquad q = 1, \cdots, p; \quad \theta_0 = 1; \quad 0 \le \theta_q \le 1 \tag{8}$$

Using the above equations the values of $\alpha_n^{(p)}$ can be found as

$$\alpha_n^{(p)} = \left(\frac{\Delta t^{p-2}}{p-2!}\theta_{p-2}\mathbf{M} + \frac{\Delta t^{p-1}}{p-1!}\theta_{p-1}\mathbf{C} + \frac{\Delta t^p}{p!}\theta_p\mathbf{K}\right)^{-1} (\bar{\mathbf{f}} - \mathbf{M}\tilde{\ddot{\mathbf{x}}}_{n+1} - \mathbf{C}\tilde{\dot{\mathbf{x}}}_{n+1} - \mathbf{K}\tilde{\mathbf{x}}_{n+1}) \tag{9}$$

where

$$\begin{aligned}
\tilde{\mathbf{x}}_{n+1} &= \sum_{q=0}^{p-1} \overset{q}{\mathbf{x}}_n \frac{\Delta t^q}{q!}\theta_q \\
\tilde{\dot{\mathbf{x}}}_{n+1} &= \sum_{q=1}^{p-1} \overset{q}{\mathbf{x}}_n \frac{\Delta t^{q-1}}{q-1!}\theta_{q-1} \\
\tilde{\ddot{\mathbf{x}}}_{n+1} &= \sum_{q=2}^{p-1} \overset{q}{\mathbf{x}}_n \frac{\Delta t^{q-2}}{q-2!}\theta_{q-2}
\end{aligned} \tag{10}$$

and assuming that the load vector is available at steps n, $n+1$ *etc.* one may assume linear variation of $\mathbf{f}$ during the time interval and use

$$\bar{\mathbf{f}} = \theta_1 \mathbf{f}_{n+1} + (1 - \theta_1)\mathbf{f}_n \tag{11}$$

This should, however, be used with care as such assumptions can lead to the change in the frequency content of the original signal (Basu, Pankaj and Kumar, 1992; Pankaj, Kumar and Basu, 1994).

Algorithm for SS22 and SS32

A detailed implementation algorithm which is a slightly modified form of that suggested by Barthwal (1992) for SS22 and SS32 algorithms is now considered

(1) Begin predictor phase
if $p = 2$ *then*
$\hat{\mathbf{x}}_{n+1} = \mathbf{x}_n + \Delta t \dot{\mathbf{x}}_n$
$\hat{\dot{\mathbf{x}}}_{n+1} = \dot{\mathbf{x}}_n$
$\tilde{\mathbf{x}}_{n+1} = \mathbf{x}_n + \Delta t \dot{\mathbf{x}}_n \theta_1$
$\tilde{\dot{\mathbf{x}}}_{n+1} = \dot{\mathbf{x}}_n$
else

$\hat{\mathbf{x}}_{n+1} = \mathbf{x}_n + \Delta t\dot{\mathbf{x}}_n + (\Delta t^2/2)\ddot{\mathbf{x}}_n$
$\hat{\dot{\mathbf{x}}}_{n+1} = \dot{\mathbf{x}}_n + \Delta t\ddot{\mathbf{x}}_n$
$\hat{\ddot{\mathbf{x}}}_{n+1} = \ddot{\mathbf{x}}_n$
$\tilde{\mathbf{x}}_{n+1} = \mathbf{x}_n + \Delta t\dot{\mathbf{x}}_n\theta_1 + \ddot{\mathbf{x}}_n(\Delta t^2/2)\theta_2$
$\tilde{\dot{\mathbf{x}}}_{n+1} = \dot{\mathbf{x}}_n + \ddot{\mathbf{x}}_n\Delta t\theta_1$
$\tilde{\ddot{\mathbf{x}}}_{n+1} = \ddot{\mathbf{x}}_n$
endif

(2) Form effective stiffness matrix $\mathbf{K}^*$
if $p = 2$ *then*
$\mathbf{K}^* = \mathbf{M}\theta_0 + \Delta t\theta_1\mathbf{C} + (\Delta t^2/2)\theta_3\mathbf{K}$
else
$\mathbf{K}^* = \mathbf{M}\Delta t\theta_1 + (\Delta t^2/2)\theta_2\mathbf{C} + (\Delta t^3/6)\theta_3\mathbf{K}$
endif

(3) Define
$\bar{\mathbf{f}} = \theta_1\mathbf{f}_{n+1} + (1-\theta_1)\mathbf{f}_n$

(4) Compute the effective load vector
if $p = 2$ *then*
$\mathbf{f}^* = \bar{\mathbf{f}} - \mathbf{C}\tilde{\dot{\mathbf{x}}}_{n+1} - \mathbf{K}\tilde{\mathbf{x}}_{n+1}$
else
$\mathbf{f}^* = \bar{\mathbf{f}} - \mathbf{M}\tilde{\ddot{\mathbf{x}}}_{n+1} - \mathbf{C}\tilde{\dot{\mathbf{x}}}_{n+1} - \mathbf{K}\tilde{\mathbf{x}}_{n+1}$
endif

(5) Set iteration counter $i = 1$ and begin iterating

(6) Evaluate residual forces
if $i = 1$ *then*
$\mathbf{x}^i_{n+1} = \hat{\mathbf{x}}_{n+1}$
$\dot{\mathbf{x}}^i_{n+1} = \hat{\dot{\mathbf{x}}}_{n+1}$
if $p = 3$ *then*
$\ddot{\mathbf{x}}^i_{n+1} = \hat{\ddot{\mathbf{x}}}_{n+1}$
endif
endif
if $p = 2$ *then*
$\psi_i = \mathbf{K}^*\hat{\mathbf{x}}_{n+1} + \mathbf{f}^*(\Delta t^2/2)/\mathbf{K}^*\mathbf{x}^i_{n+1}$
else
$\psi_i = \mathbf{K}^*\hat{\mathbf{x}}_{n+1} + \mathbf{f}^*(\Delta t^3/6) - \mathbf{K}^*\mathbf{x}^i_{n+1}$
endif

(7) Solve for incremental displacements $\Delta\mathbf{x}^i$
$\mathbf{K}^*\Delta\mathbf{x}^i = \psi_i$

(8) Begin the corrector phase

$\mathbf{x}_{n+1}^{i+1} = \mathbf{x}_{n+1}^{i} + \Delta\mathbf{x}^{i}$
if $(p = 2)$ *then*
$\dot{\mathbf{x}}_{n+1}^{i+1} = \hat{\dot{\mathbf{x}}}_{n+1} + (\mathbf{x}_{n+1}^{i+1} - \hat{\mathbf{x}}_{n+1})(2/\Delta t)$
else
$\dot{\mathbf{x}}_{n+1}^{i+1} = \hat{\dot{\mathbf{x}}}_{n+1} + (\mathbf{x}_{n+1}^{i+1} - \hat{\mathbf{x}}_{n+1})(3/\Delta t)$
$\ddot{\mathbf{x}}_{n+1}^{i+1} = \hat{\ddot{\mathbf{x}}}_{n+1} + (\mathbf{x}_{n+1}^{i+1} - \hat{\mathbf{x}}_{n+1})(6/\Delta t^2)$
endif

(9) *if* ($\Delta\mathbf{x}^{i}$ and/or $\psi >$ specified tolerences) *then*
apply iteration counter increment and *go to* (6)
else
go to (10)
endif

(10) *if* $(p = 2)$ *then*
$\mathbf{x}_{n+1} = \mathbf{x}_{n+1}^{i+1}$
$\dot{\mathbf{x}}_{n+1} = \dot{\mathbf{x}}_{n+1}^{i+1}$
else
$\mathbf{x}_{n+1} = \mathbf{x}_{n+1}^{i+1}$
$\dot{\mathbf{x}}_{n+1} = \dot{\mathbf{x}}_{n+1}^{i+1}$
$\ddot{\mathbf{x}}_{n+1} = \ddot{\mathbf{x}}_{n+1}^{i+1}$
endif

(11) Set $n = n + 1$ and begin next step

MODIFICATIONS FOR NONLINEAR ANALYSIS

In this paper the nonlinear problems discussed are confined to those arising out of elastoplastic material behaviour. Thus nonlinearity is confined to a nonlinear stiffness matrix which in turn is dependent on displacements. So mere replacement of $\mathbf{K}$ by the elastoplastic stiffness matrix $\mathbf{K}^{ep}$ modifies the above algorithms for such an analysis (Owen and Hinton, 1980). When a modified Newton Raphson (mN-R) iterative procedure is used one can continue to use $\mathbf{K}$ as far as the formation of the effective stiffness matrix $\mathbf{K}^{*}$ is concerned, while taking into account the effect of elastoplastic stiffness in the evaluation of residual force vector. If the stiffness matrix is reformulated in every iteration (N-R iteration) convergence may be achieved faster *i.e.* in fewer iterations. However additional computational effort is required in the formation of stiffness matrix and its triangularisation in every iteration. Further N-R procedure suffers from a drawback that $\mathbf{K}$ may become singular for elasto-perfectly-plastic or elasto-strain softening materials.

Convergence criteria

The convergence criteria were based on the norm of incremental displacements and the norm of residual forces. Thus the solution is said to have converged if

$$\left| \frac{\left((\mathbf{x}_n^{i+1})^T \mathbf{x}_n^{i+1} - (\mathbf{x}_n^i)^T \mathbf{x}_n^i \right)}{\left((\mathbf{x}_n^1)^T \mathbf{x}_n^1 \right)} \right| \leq toler1 \tag{12}$$

and

$$\left| \frac{\left((\boldsymbol{\psi}_n^{i+1})^T \boldsymbol{\psi}_n^{i+1} - (\boldsymbol{\psi}_n^i)^T \boldsymbol{\psi}_n^i \right)}{\left((\boldsymbol{\psi}_n^1)^T \boldsymbol{\psi}_n^1 \right)} \right| \leq toler2 \tag{13}$$

*toler*1 and *toler*2 are the specified tolerances and were kept at 0.001. Superscripts denote iteration number.

EXAMPLE PROBLEMS

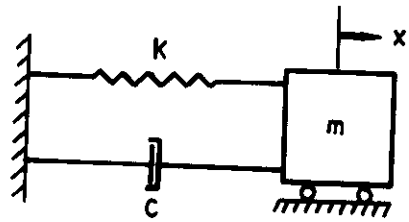
The results of some particular examples of SS22, SS32, GN22 and GN32 are presented. Values of the parameters θ_q in SS*pj* and β_q in GN*pj* are so chosen that the algorithms are *single step equivalents* of well known methods. Such equivalence has been established for SSpj algorithms by Zienkiewicz *et al.* (1984) and Zienkiewicz and Katona (1985). Here we consider the equivalent Houbolt and Wilson θ methods. The Houbolt method is originally a multistep method. For the SS and GN methods outlined in the previous sections it has been shown that SS22 and GN22 are equivalent when $\beta_1 = \beta_2 = \theta_1 = \theta_2 = 0.5$ (Zienkiewicz *et al.*, 1984). GN32 and SS32 become equivalent to the Houbolt method when $\beta_1 = \theta_1 = 2$, $\beta_2 = \theta_2 = 11/3$, $\beta_3 = \theta_3 = 6$ and to the Wilson θ method when $\beta_1 = \theta_1 = \theta_w$, $\beta_2 = \theta_2 = \theta_w^2$, $\beta_3 = \theta_3 = \theta_w^3$ (Zienkiewicz *et al.*, 1984; Zienkiewicz and Katona, 1985). The Wilson θ method is unconditionally stable when $\theta_w \geq 1.366$ (Bathe, 1982).

The single step equivalents of GN32 and SS32 have not been sufficiently tested. While some tests with SS32 have been made (Penery and Wood, 1985) the authors were unable to find similar research with regard to GN32. The reason, perhaps, is that GN32 is difficult to use when the initial conditions are other than zero. This is due to the fact that the method requires initial value of the third derivative of displacement for which no expression is available.

Some linear and nonlinear problems were solved to study the performance of these algorithms. The problems and the results are discussed in the following subsections.

Problem 1 : Free vibration due to initial displacement

In this problem the mass of the system shown in Fig. 2 is displaced by unity and the system permitted to vibrate freely due to this initial condition. In addition the system



$$K = 4100\,N/mm$$
$$m = 100\,Kg$$

Figure 2: Damped SDF system.

is assumed to be undamped. This problem was also considered by Bathe (1982) to study the performance of various methods. The time step Δt chosen is $T/20$. A Single Degree Freedom (SDF) system as shown in Fig. 2 forms the basis for this and some other problems that follow.

It can be seen from Figs. 3 to 5 for this free vibration problem, for the choice of

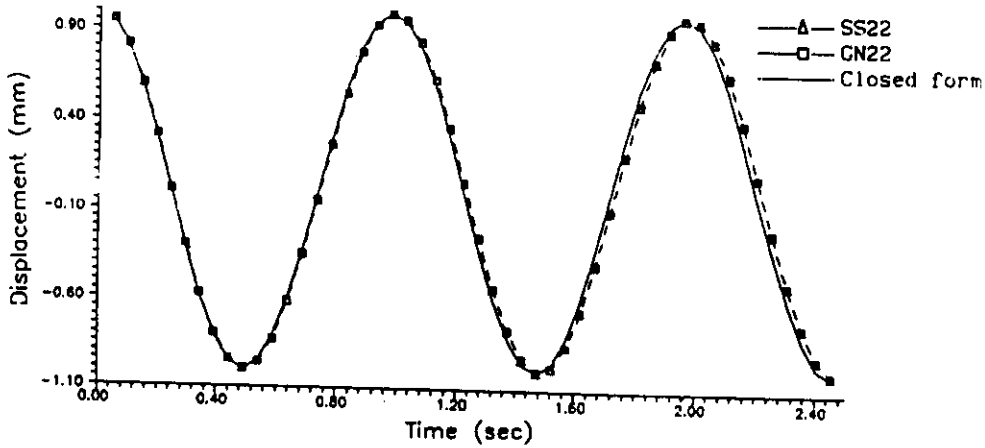


Figure 3: Free vibration response under unit initial dispalcement. Comparision of solutions by SS22, GN22 with closed form (Problem 1).

Δt made, that the results of SS22 and GN22; SS32 and GN32 (Houbolt case); and SS32 and GN32 (Wilson θ case) are identical. The Period Elongation (PE) and Amplitude Decay (AD) is the maximum for the Houbolt Cases of SS32 and GN32 (Fig. 4) while it is the least for SS22 and GN22 algorithms. In fact SS22 and GN22 exhibit only PE and no AD. For this free vibration problem, no difference is seen between similar SS and GN algorithms.

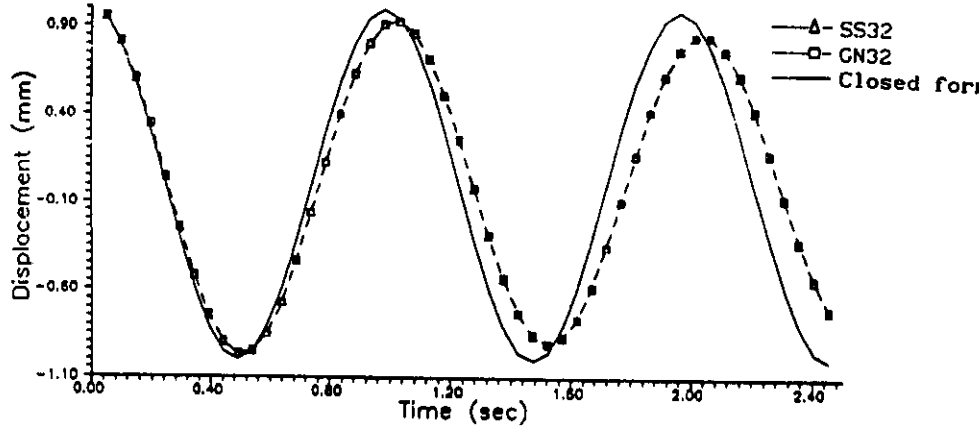


Figure 4: Free vibration response under unit initial dispalcement. Comparision of solutions by SS32, GN32 (Houbolt case) with closed form (Problem 1).

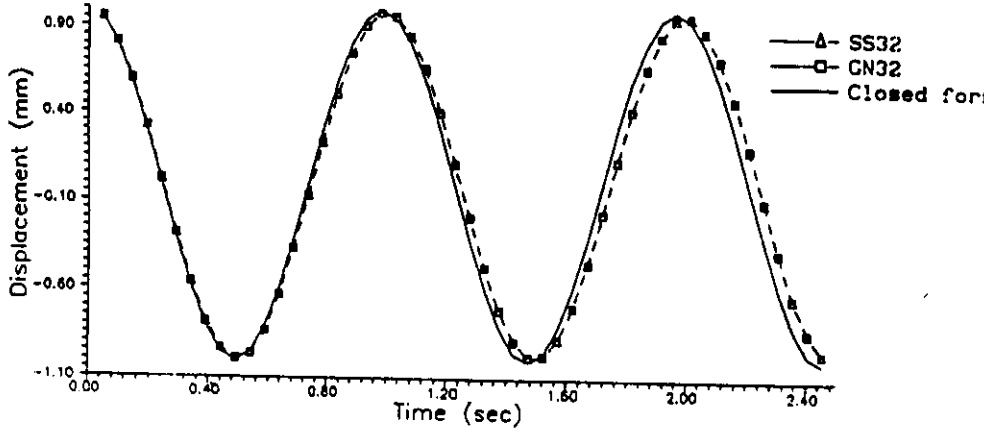


Figure 5: Free vibration response under unit initial dispalcement. Comparision of solutions by SS32, GN32 (Wilson θ case) with closed form (Problem 1).

Problem 2 : Sinusoidal base excitation

The base of the system shown in Fig. 2 is excited by a sinusoidal displacement function $Y = Y_0 \sin \omega t$ with $Y_0 = 1.2$ *mm* and $\omega = 4$ *rad/sec*. The system is assumed to be undamped and Δt is chosen to be the same as before.

Figure 6 once again indicates identical results from SS22 and GN22 algorithms.

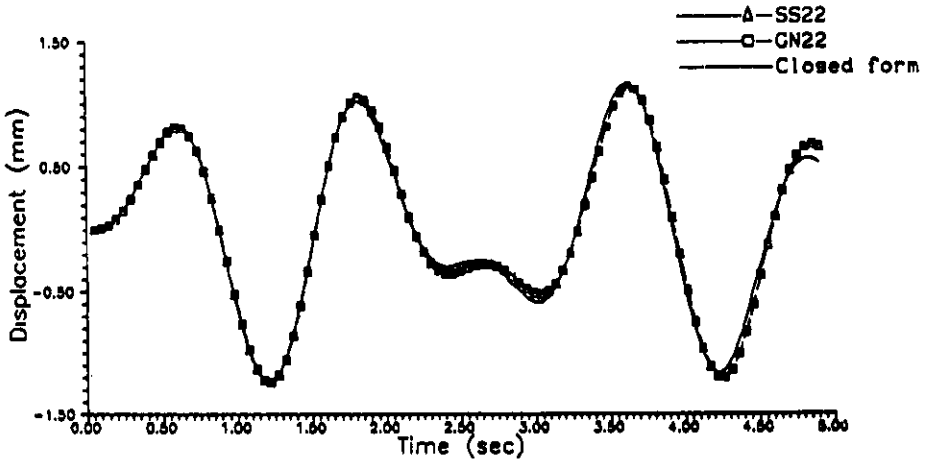


Figure 6: Comparsion of solutions by SS22, GN22 and closed form under sinusoidal base excitation (Problem 2).

The Houbolt case of SS32 and GN32 deviates considerably from the closed form solution (Fig. 7). The maximum difference between the closed form and Houbolt equivalent of SS32/GN32 (Fig. 7) was seen to be almost three times the maximum difference between closed form and SS22/GN22 (Fig. 6). In the Wilson θ equivalent of SS32 and GN32 (Fig. 8) the numerical solution deviates from the closed form solution and the maximum difference is greater than the maximum difference between closed form and SS22/GN22 cases. The SS22 and GN22 algorithms gave identical response. The response obtained from the Houbolt equivalent of SS32 and GN32 was close. Similarly the response obtained from the Wilson θ equivalent of SS32 and GN32 was close. One would have expected that if both the families were simulating the Houbolt and Wilson θ equivalents the response would be identical. The reason for this not being so, apparently, lies in the calculation of the equivalent load vector $\bar{\mathbf{f}}$ used in the SS algorithm, which needs to be approximated using a kind of mean value between the actual discrete load points supplied. This is not so for the GN algorithm which uses values only at end points.

Problem 3 : Sudden load reversal

The mass of the system of Fig. 2 is subjected to a force as shown in Fig. 9 in which

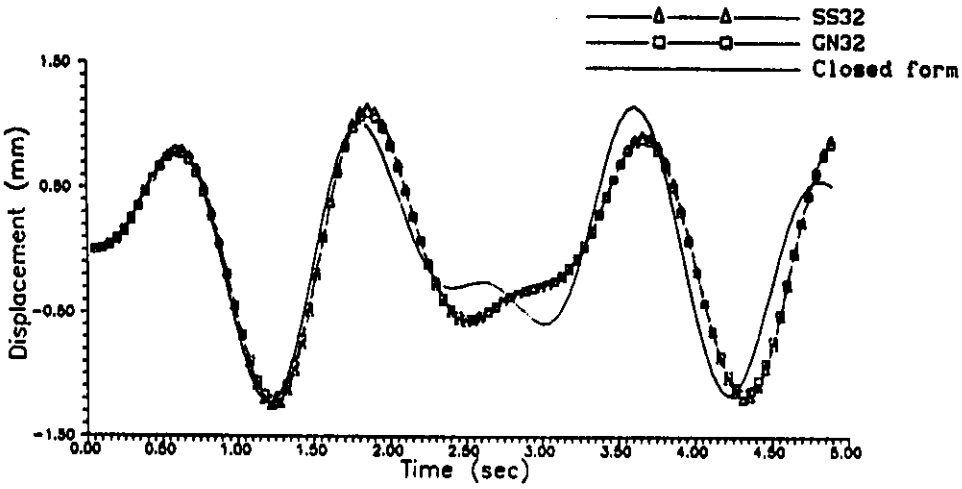


Figure 7: Comparsion of solutions by SS32, GN32 (Houbolt case) and closed form under sinusoidal base excitation (Problem 2).

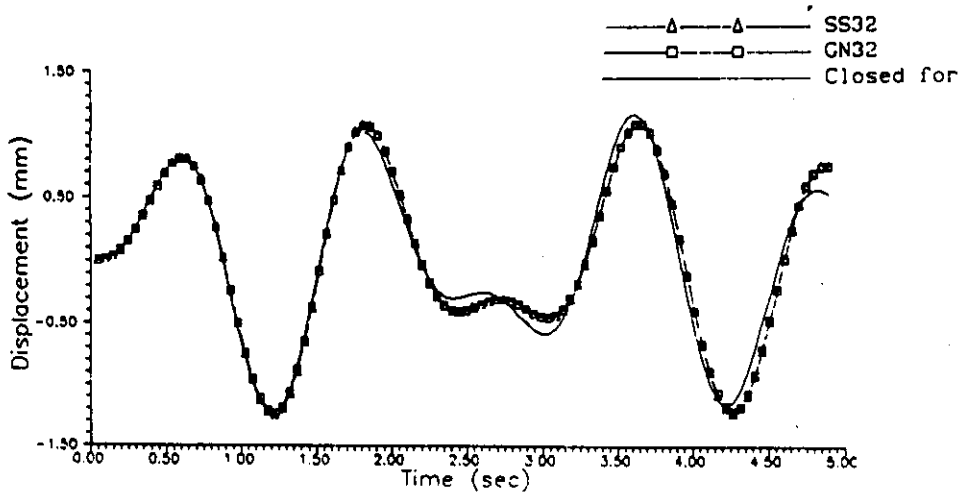


Figure 8: Comparsion of solutions by SS32, GN32 (Wilson θ case) and closed form under sinusoidal base excitation (Problem 2).

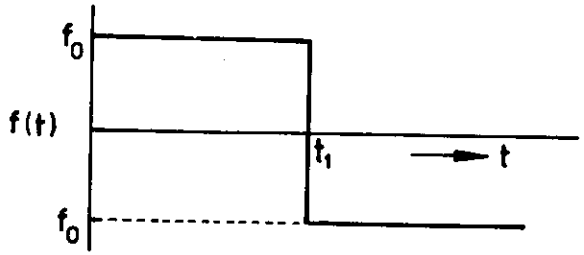


Figure 9: Suddenly reversing force (Problem 3).

$f_0 = 2000\ N$, $t_1 = 4.90\ sec$. Damping is assumed to be 20% of critical. This problem is of interest as it amply illustrates the need for time step changes during the analysis.

In this study the time step was not changed during a single analysis, however, two sets of analyses with different time steps were done. These analyses adequately illustrate that time step may be required to be changed depending on the loading and the response history. The time steps chosen were $T/10$ and $T/50$ where T is the natural period of the system. Thus two sets of curves were drawn for this problem. Each set is a combination of curves of solutions obtained from SS22, GN22, SS32, GN32 (Houbolt case), SS32, GN32 (Wilson θ case) and closed form (Figs. 10 and 11). From these figures it can be seen that the conclusions drawn earlier regarding the performance of various methods remain unchanged. Clearly the results improve when the time step is reduced. However, the larger time step also yields an accurate response in the steady state region (which in this problem is between 4.0 and 4.9 *sec* and then again beyond 9.3 *sec*). Clearly, therefore, as the frequency of oscillations reduces, the time step size can be increased and when it increases the time step needs to be reduced. This feature is easy to incorporate in single step procedures as compared to multistep algorithms, provided suitable indicators to signal such a change can be devised. Some indicators based on error estimates have been devised by some investigators (Zienkiewicz *et al.*, 1984).

Problem 4 : Step function load on nonlinear systems

The structure considered for this problem is once again the single degree freedom system of Fig. 2. A step finction load of 2000 N was applied on the mass. The system was assumed to be undamped. Different nonlinear characteristics of the spring used are shown in Fig. 12. Figure 12(a) shows elastic perfectly plastic spring, while Figs. 12(b) and (c) show strain hardening and softening springs respectively.

The nonlinear solution algorithm employed made use of the initial stiffness iterative procedure wherein the effective stiffness was kept unchanged in the solution of iterative/incremental displacements. The affect of the nonlinearity is incorporated through the calculation of the residual force vector. Clearly if the regular N-R iterative

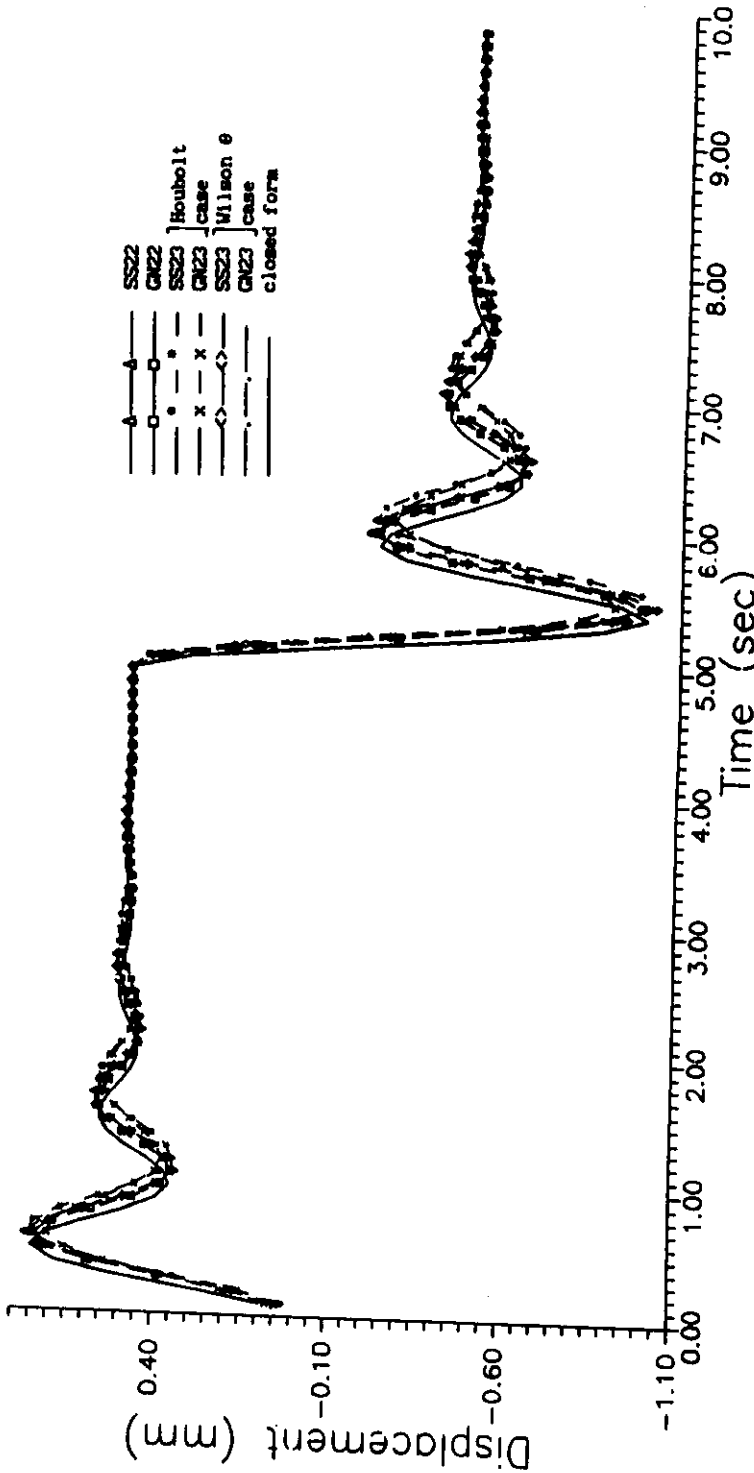


Figure 10: Sudden load reversal, $\Delta t = T/10$, comparision of solutions by SS22, GN22; SS23, GN32 (Houbolt case); SS23, GN23 (Wilson θ case) with closed form (Problem 3).

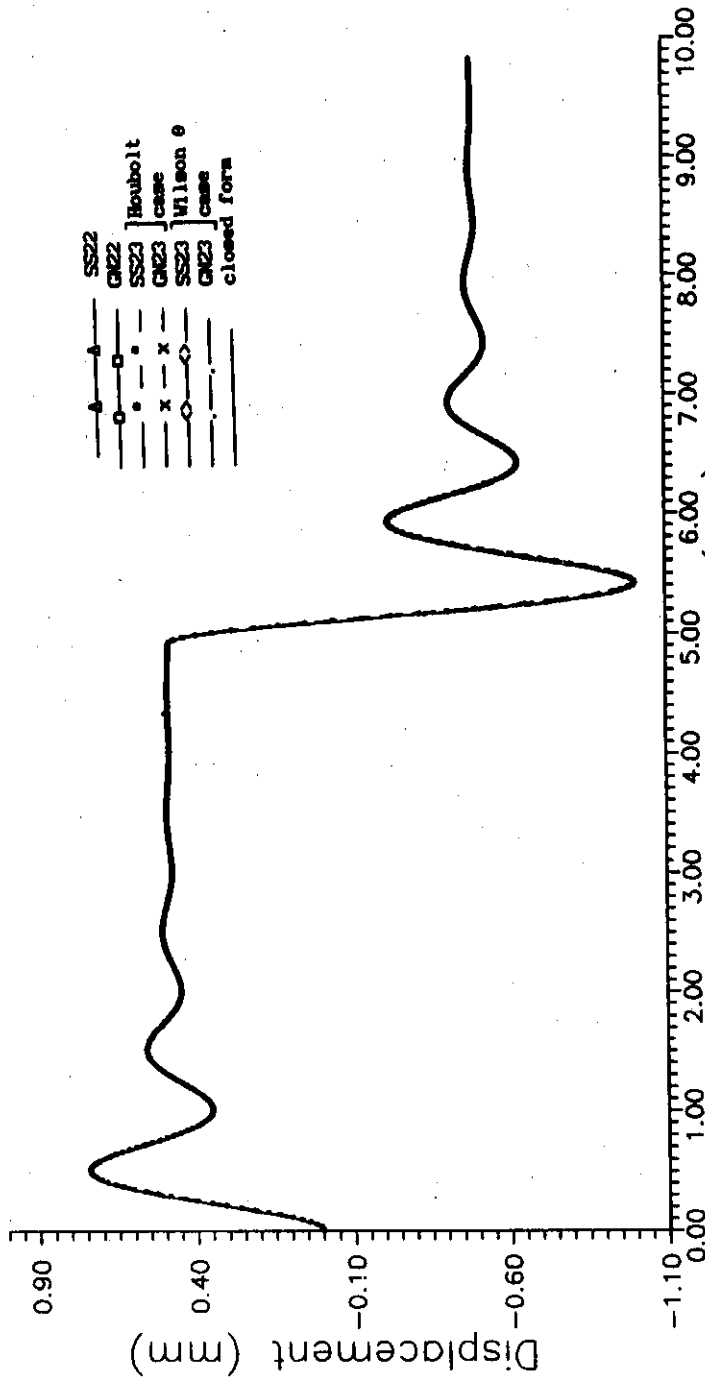


Figure 11: Sudden load reversal, $\Delta t = T/50$, comparision of solutions by SS22, GN22; SS23, GN23 (Houbolt case); SS23, GN23 (Wilson θ case) with closed form (Problem 3).

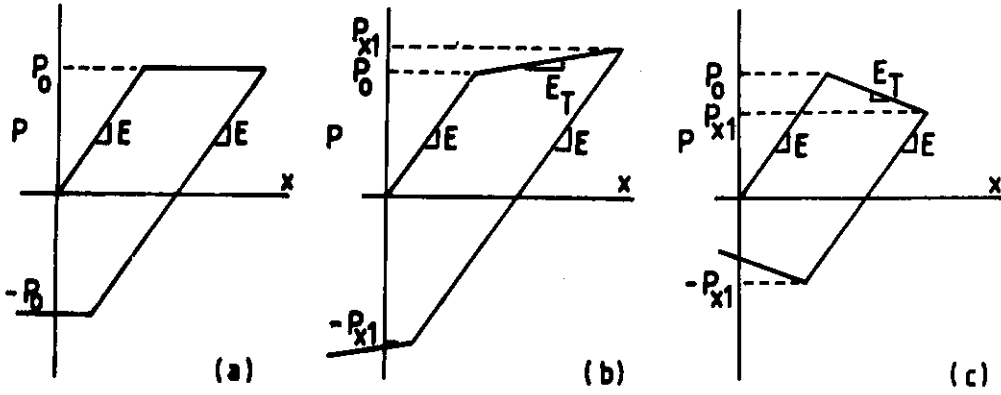


Figure 12: (a) Elastic – perfectly plastic (b) Elastic – strain hardening plastic (c) Elastic – strain softening plastic.

process were employed the stiffness matrix would not be positive definite for perfectly plastic and softening cases. A total residual strategy for dynamic analysis was devised, similar to the one used for static analysis by Bicanic and Pankaj (1990).

The first series of analyses was conducted with an elastic perfectly plastic spring with an yield force $P_0 = 3280\,N$. For such a system an exact (closed form) displacement response was found and plotted along with GN22 and GN32 (Wilson θ and Houbolt cases). For each case a $\Delta t = 0.05\,sec$ was employed. The response is shown in Fig. 13. Once again it can be seen that GN22 is the closest to the exact solution, while Houbolt equivalent of the of GN32 is the farthest. Thus a simple nonlinear problem of this kind illustrates the efficacy and accuracy of the GN22 algorithm.

The authors encountered considerable difficulty in using the SS algorithm for nonlinear problems. This algorithm uses "mean predictor values" ($\tilde{\mathbf{x}}$ *etc.*) which are used in the evaluation of the effective load vector (Step 5). It appears that for nonlinear problems these mean predictor values would need to be revised with every iteration. In fact while using the SS algorithm with nonlinear problems of the kind being discussed we are solving the differential equation

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{P}(\mathbf{x}) = \tilde{\mathbf{f}} \tag{14}$$

where $\mathbf{x}$ is a sort of an average of displacement between steps n and $n+1$. $\mathbf{P}(\mathbf{x})$ is the restoring force at a value of $\mathbf{K}$ (say $\tilde{\mathbf{K}}$) at this $\mathbf{x}$. Thus

$$\mathbf{P} = \mathbf{P}(\mathbf{x}_n) + \tilde{\mathbf{K}}\Delta\mathbf{x} \tag{15}$$

In other words appropriate interpolation for $\mathbf{P}(\mathbf{x})$ needs to be made. Moreover, the residual forces at the end of the step *i.e.* $\mathbf{P}(\mathbf{x}_{n+1})$ also need to be evaluated for use in the next time step. This sort of multiple evaluation of the residual forces complicates the SS procedure for nonlinear analysis. Therefore, the solution with this family of algorithms was abandoned.

Using the GN22 algorithm, which was seen to perform the best for the perfectly plastic case, a second series of analyses was carried out. The displacement response under step function loading was evaluated assuming the spring to be (a) elastic, (b) perfectly plastic (Fig. 12(a)), (c) strain hardening (Fig. 12(b)) with slope of the hardening branch $E_T = 410\,N/mm$, and (d) strain softening (Fig. 12(c)) with $E_T = -410\,N/mm$. For the nonlinear cases initial yield force $P_0 = 3280\,N$ was assumed.

The displacement response is shown in Fig. 14. It can be seen that the irreversible

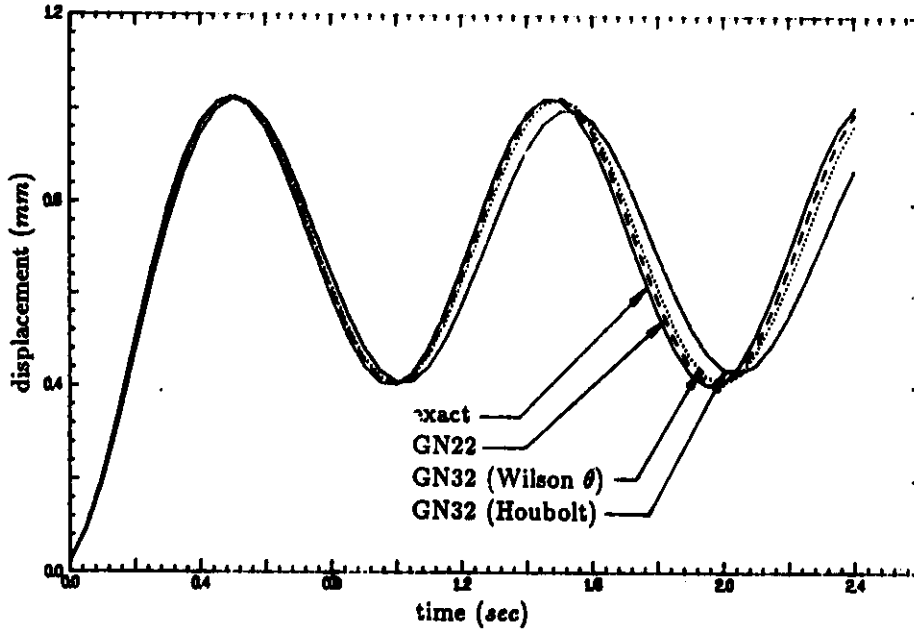


Figure 13: Response of SDF system with elastic – perfectly plastic spring to step function loading (Problem 4).

plastic displacement increases as the spring characteristics are changed from elastic to strain hardening to perfectly plastic to strain softening. It would appear that the period of oscillation would increase in the same sequence (Zienkiewicz and Wood, 1986). A Fourier analysis of the frequency content, however, reveals that there is no change in the predominant frequency of vibration as the spring characteristics change (Fig. 15). In each case there is a zero frequency component (ZFC) which is due to the mass not vibrating about the zero displacement position. This component is the highest for the strain softening case and zero for the elastic case. The analysis shows that strain softening which has been successfully used for static analysis can also be used for dynamic analysis.

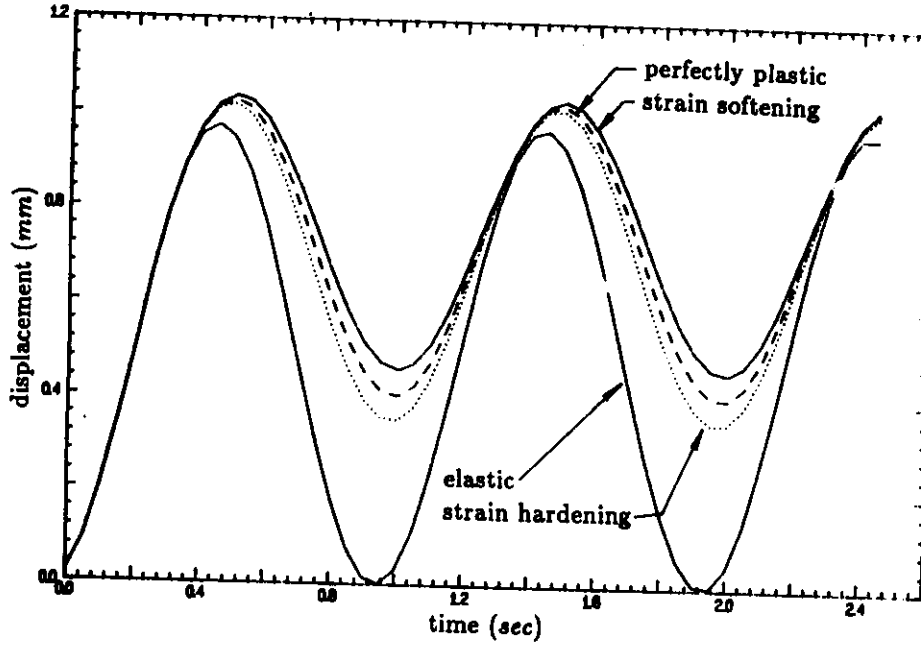


Figure 14: Response of SDF systems with different spring characteristics to step function loading using GN22 algorithm (Problem 4).

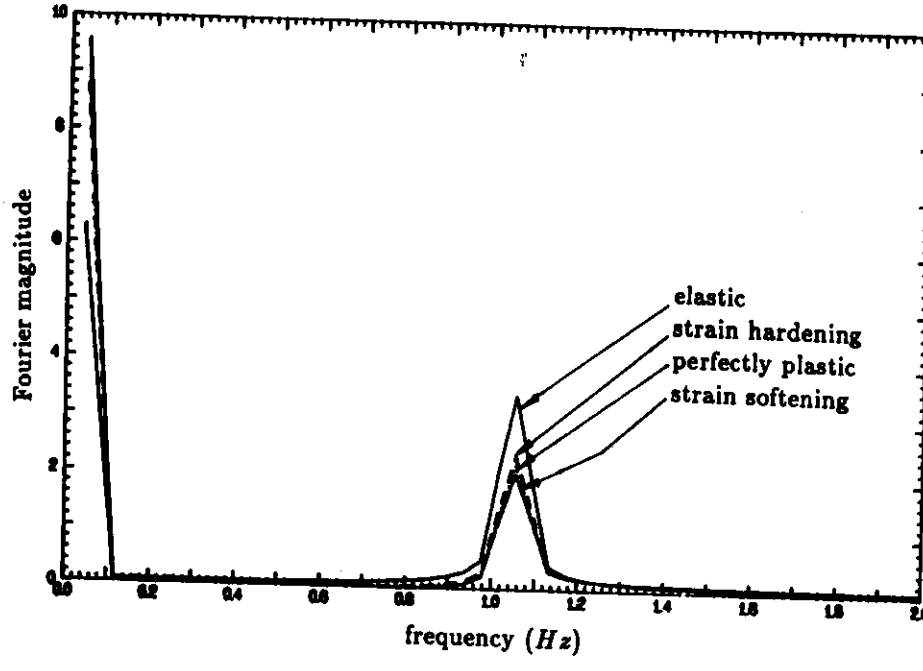


Figure 15: Fourier magnitude of displacement response of a SDF system with different spring characteristics under step function load (Problem 4).

Problem 5 : Seismic response of nonlinear SDF systems

The base of the single degree freedom (SDF) system of Fig. 2 was subjected to an actual earthquake acceleration history. The corrected accelerogram of the Uttarkashi earthquake of October 20, 1991 obtained at 30.738N and 78.792E (Earthquake Engineering Studies, 1993) was used for the purpose. A constant viscous damping of 5% of critical was considered and the spring was assumed to be (a) elastic, (b) perfectly plastic, and (c) strain softening. For the strain softening situation $E_T = -410\,N/mm$ was assumed. The response was computed using the GN22 algorithm with $\Delta t = 0.02\,sec$. In the first instance the yield force value $P_0 = 2.05 \times 10^5\,N$ was assumed. The displacement response is shown in Fig. 16. It can be seen that for nonlinear cases the mass does

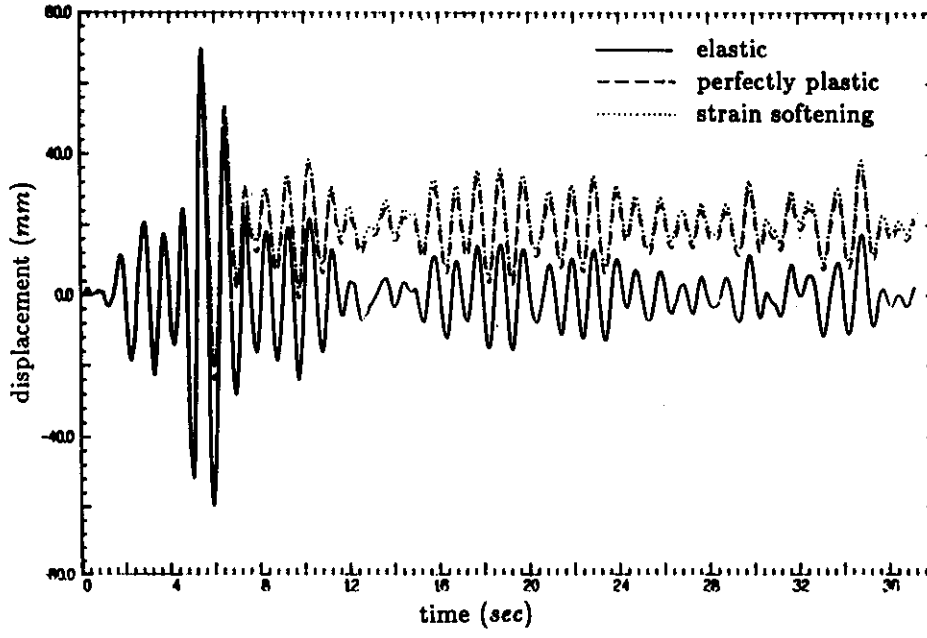


Figure 16: Displacement response of SDF system with different spring characteristics (with $P_0 = 2.05 \times 10^5\ N$ for nonlinear cases) under Uttarkashi earthquake excitation (Problem 5).

not vibrate about the zero displacement position. The Fourier analysis (Fig. 17) of the response indicates that, in general, the predominant frequency content does not change with the change in spring characteristics although some low frequency components appear to have been added to the response. Due to the mass finding new mean positions to vibrate about a zero frequency component is added to the response (Fig. 17). These changes of the mean position take place with the building up of the plastic displacement, which happens in a short time (Fig. 18) and remains constant thereafter. The maximum plastic displacement for this example was found to be $22.43\,mm$ and $25.03\,mm$ for perfectly plastic and softening cases respectively.

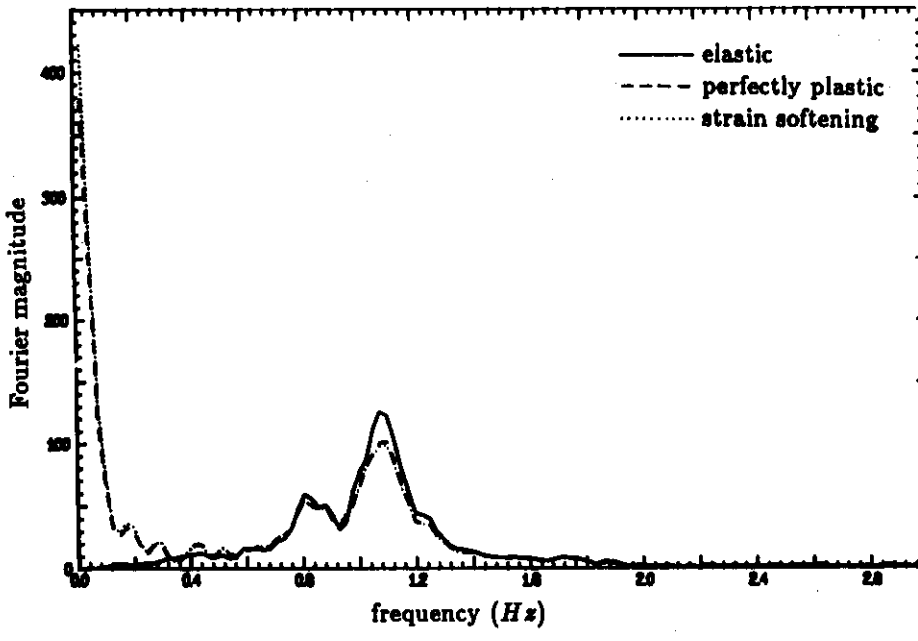


Figure 17: Fourier magnitude of displacement response of SDF system with different spring characterstics (with $P_0 = 2.05 \times 10^5$ N for nonlinear cases) under Uttarkashi earthquake excitation (Problem 5).

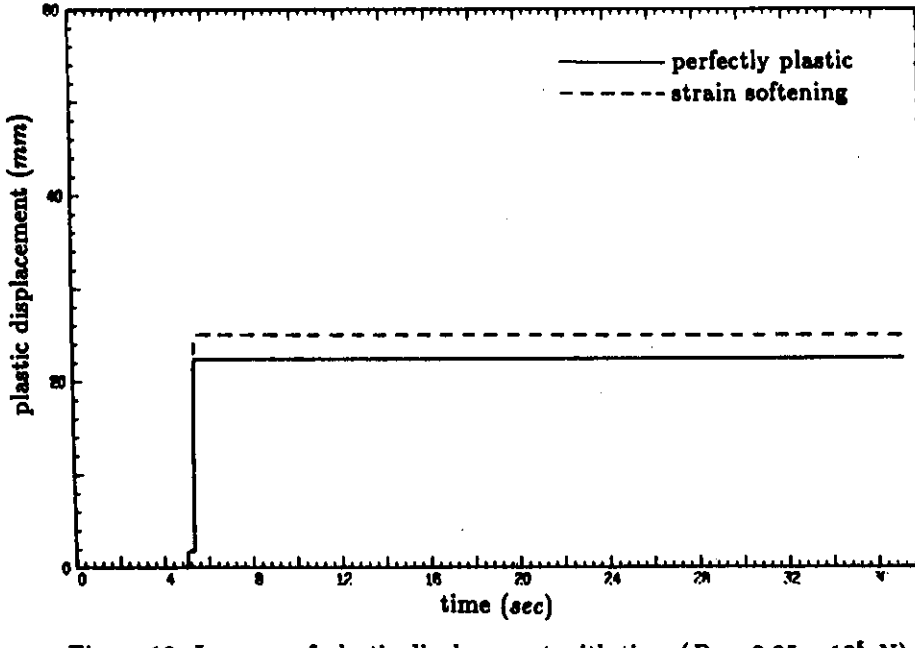


Figure 18: Increase of plastic displacement with time ($P_0 = 2.05 \times 10^5$ N)

CONCLUSIONS

Stepwise procedures for SS and GN family of algorithms were developed in the predictor corrector form. From examples studied it appears that procedures with $p = 2$ and constants $\beta_1 = \beta_2 = \theta_1 = \theta_2 = 0.5$, which convert the SS and GN algorithms to Newmark's unconditionally stable algorithm are superior to the Houbolt and Wilson θ equivalents with $p = 3$. Theoretically the SS procedure is better as it requires one less starting condition as compared to the GN procedure. Thus when $p = 3$ the GN procedure the third derivative of x as a starting condition which is not readily available. Moreover, in SSpj the last derivative required during analysis is $\overset{p-1}{\mathbf{x}}_n$ while in GNpj the last derivative required is $\overset{p}{\mathbf{x}}_n$. However, the SS procedure requires the equivalent load vector to be approximated using a kind of mean value between actual discrete load points. This is not so for the GN algorithm which uses values only at the end points. Thus the SS algorithm uses an additional approximation with regard to the variation of load between two discrete load points. This poses difficulties in the computation of the "effective load vector" in elastoplastic problems. So for nonlinear problems the GN family is seen to be superior.

Elastoplastic stiffness characteristics do not alter the predominant frequency response, although a zero frequency component gets incorporated due to plastic displacement. Strain softening can be successfully employed in dynamic problems.

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