Paper No. 138, Bulletin I.S.E.T., Vol. 10, No. 4, pp. 145-153, December 1973.

FREE VIBRATION ANALYSIS OF HORIZONTALLY CURVED GIRDERS

S. K. THAKKAR[1] AND M. QAMARUDDIN[2]

INTRODUCTION

There is an extensive literature on the static analysis of horizontally curved beam or girders, but little has been done on the dynamic analysis of such members. Vibration of curved girder bridges under the action of seismic forces or moving loads and vibration of arches out of their plane are some examples where this problem assumes importance. Recently the problem of dynamic analysis of such members has started receiving attention[3,4,5]. Closed form solutions are obtained for circular curved girders having uniform cross-section[1]. Closed form solutions are also derived for fundamental frequency of curved girders in the form of circle, cycoid and parabola of uniform cross-section using Rayleigh-Ritz method[6]. Closed form expressions for higher modes of vibration becomes much involved for circular and other shapes of girders. In this paper a numerical method has been presented for free vibration analysis of any shape and any distribution of mass and stiffness of girders. The method consists in obtaining a flexibility coefficient matrix for an equivalent lumped mass mathematical model of the girder[2]. The natural frequencies are obtained by matrix iteration that yields the frequencies and modes in ascending order, hence the method is suitable for computing response to any dynamic load. The present analysis considers both bending and torsional vibrations.

METHOD OF ANALYSIS

The continuous curved girder is replaced by discrete lumped mass system. The masses are connected to each other by massless flexible springs. The following assumptions are made:

1. Only the out of plane displacements are considered and taken as small.
2. All stresses and deformations are within the elastic limits of the material.
3. Both bending and torsional deformations and considered. The effect of shearing deformation, rotatory inertia and axial forces are neglected.
4. Each mass is considered to have degree of freedom in translation and twist.

Consider two lumped masses m_1 and m_2 as shown in Fig. 1. The subscripts v and t

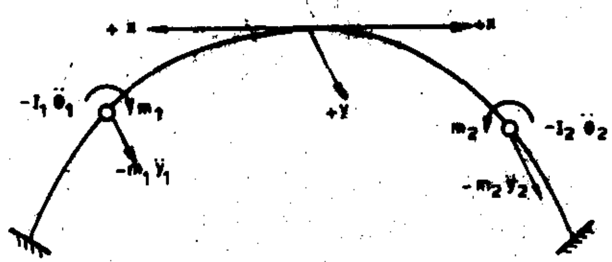

Fig. 1, Lumped mass system of curved girder

1 Reader in Structural Dynamics, School of Research and Training in Earthquake Engineering, University of Roorkee, Roorkee, (U. P.), India.
2 Lecturer in Civil Engineering Department, Aligarh Muslim University, Aligarh (U. P.), India.

are used to distinguish between vertical and rotational displacements and loads respectively. For instance δ_{ij}^{vt} is the vertical displacement at i caused by unit torsional moment at j. The equation of motion for free undamped vibration can be written as:

$$y_1 = -m_1\ddot{y}_1\,\delta_{11}^{vv} - m_2\ddot{y}_2\,\delta_{12}^{vv} - I_1\ddot{\theta}_1\,\delta_{11}^{vt} - I_2\ddot{\theta}_2\,\delta_{12}^{vt}$$

$$y_2 = -m_1\ddot{y}_1\,\delta_{21}^{vv} - m_2\ddot{y}_2\,\delta_{22}^{vv} - I_1\ddot{\theta}_1\,\delta_{21}^{vt} - I_2\ddot{\theta}_2\,\delta_{22}^{vt}$$

$$\theta_1 = -m_1\ddot{y}_1\,\delta_{11}^{tv} - m_2\ddot{y}_2\,\delta_{12}^{tv} - I_1\ddot{\theta}_1\,\delta_{11}^{tt} - I_2\ddot{\theta}_2\,\delta_{12}^{tt}$$

$$\theta_2 = -m_1\ddot{y}_1\,\delta_{21}^{tv} - m_2\ddot{y}_2\,\delta_{22}^{tv} - I_1\ddot{\theta}_1\,\delta_{21}^{tt} - I_2\ddot{\theta}_2\,\delta_{22}^{tt} \qquad \ldots 1$$

where y_1 and θ_1 denote the component of vertical deflection and twist respectively for mass m_1 in the positive direction, $\ddot{y}_1$ and $\ddot{\theta}_1$ represent the component of accelerations of mass m_1 in the positive direction. y_2, θ_2, $\ddot{y}_2$ and $\ddot{\theta}_2$ represent the corresponding quantities for mass m_2. I_1 and I_2 are torsional constant for the cross-section at the masses m_1 and m_2 respectively. By assuming the solution as,

$$y_1 = A_1 \text{ Sin pt}\ ;\quad y_2 = A_2 \text{ Sin pt}$$
$$\theta_1 = A_3 \text{ Sin pt}\ ;\quad \theta_2 = A_4 \text{ Sin pt} \qquad \ldots 2$$

and substituting in Eq. 1 and simplifying, the following equations are obtained,

$$\left(m_1\delta_{11}^{vv} - \frac{1}{p^2}\right)A_1 + m_2\,\delta_{12}^{vv}A_2 + I_1\,\delta_{11}^{vt}A_3 + I_2\,\delta_{12}^{vt}A_4 = 0$$

$$m_1\delta_{21}^{vv}A_1 + \left(m_2\delta_{22}^{vv} - \frac{1}{p^2}\right)A_2 + I_1\delta_{21}^{vt}A_3 + I_2\,\delta_{22}^{vt}A_4 = 0$$

$$m_1\delta_{11}^{tv}A_1 + m_2\delta_{12}^{tv}A_2 + \left(I_1\delta_{11}^{tt} - \frac{1}{p^2}\right)A_3 + I_2\delta_{12}^{tt}A_4 = 0$$

$$m_1\delta_{21}^{tv}A_1 + m_2\delta_{21}^{tv}A_2 + I_1\delta_{21}^{tt}A_3 + \left(I_2\delta_{22}^{tt} - \frac{1}{p^2}\right)A_4 = 0 \qquad \ldots 3$$

For a non-trivial solution of Equation 3,

$$\begin{vmatrix} \left(m_1\delta_{11}^{vv} - \frac{1}{p^2}\right) & m_2\delta_{12}^{vv} & I_1\delta_{11}^{vt} & I_2\delta_{12}^{vt} \\ m_1\delta_{21}^{vv} & \left(m_2\delta_{22}^{vv} - \frac{1}{p^2}\right) & I_1\delta_{21}^{vt} & I_2\delta_{22}^{vt} \\ m_1\delta_{11}^{tv} & m_2\delta_{12}^{tv} & \left(I_1\delta_{11}^{tt} - \frac{1}{p^2}\right) & I_2\delta_{12}^{tt} \\ m_1\delta_{21}^{tv} & m_2\delta_{22}^{tv} & I_1\delta_{21}^{tt} & \left(I_2\delta_{22}^{tt} - \frac{1}{p^2}\right) \end{vmatrix} = 0 \qquad \ldots 4$$

By solving this frequency determinant, four values of p^2 can be obtained and for each value of p^2, there is a set of vectors defining the mode shapes. For the general case of n masses, the equations of motion may be expressed in the form,

$$[G][M]\ \{\ddot{U}\} + \{U\} = \{0\} \qquad \ldots 5$$

where $\{U\}$ is the column matrix consisting of vertical displacements and twists of masses; $[G]$ is a square matrix for flexibility coefficients of order 2n and $[M]$ is a diagonal matrix of order 2n. Assuming the solution of Eq. 5 to be $\{U\} = e^{ipt}\{\Phi\}$, the following equation is obtained,

$$[G]\,[M]\,\{\Phi\}=\frac{1}{p^2}\,\{\Phi\} \qquad \ldots 6$$

Equation (6) is the matrix formulation of the undamped free vibration problem in which $\frac{1}{p^2}$ is the eigen value and $\{\Phi\}$ is the eigen vector. Determination of eigen value and eigen vector yield the frequency and mode shape respectively.

Flexibility coefficients of curved girder :

The deflection at any point can be determined by considering strain energy due to flexure and torsional deformations,

$$\delta_{ij}=\frac{\partial V}{\partial W}=\int\frac{M_n}{EI}\,\frac{\partial M_n}{\partial w}\,ds+\int\frac{M_t}{GJ}\,\frac{\partial M_t}{\partial w}\,ds \qquad \ldots 7$$

in which δ_{ij} is the deflection at any point i due to unit load at j. For arbitrary shape of the girder axis and any variation of girder cross-section the direct integration of above equation is not possible. Therefore, numerical integration is restored. To get a good precision, the girder axis has been divided into thirty segments for deflection calculations. The deflections are calculated at the places where masses are concentrated. In the present analysis the masses are concentrated symmetrically with respect to axis of symmetry and they are located at the centres of segments. The advantage of symmetry and reciprocal relationship is taken in order to avoid duplication in numerical work. The frequencies and modes are computed by the matrix iteration procedure. The numerical computations are carried out on IBM 1620 computer.

RAYLEIGH-RITZ METHOD

The following expressions for potential energy V and kinetic energy T could be written for curved girder, as shown in Fig. 2.

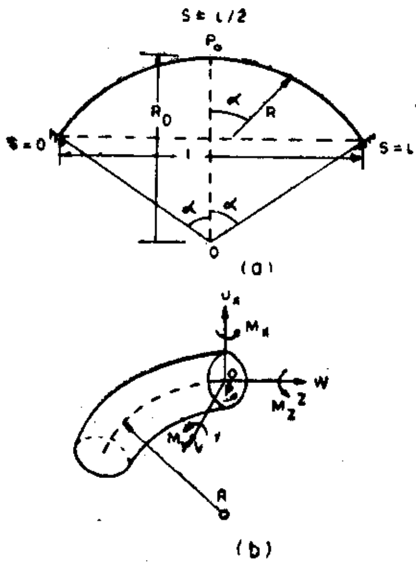

Fig. 2. Sign Convention for forces and coordinate system

$$V=\frac{EI}{2}\int_0^L\left[\left(\frac{\theta}{R}-\frac{\partial^2 y}{\partial s^2}\right)^2\right]ds+\frac{GI_p}{2}\int_0^L\left[\left(\frac{\partial\theta}{\partial s}+\frac{1}{R}\frac{\partial y}{\partial s}\right)^2\right]ds$$

$$T=\frac{\rho A_c}{2}\int_0^L\left[\left(\frac{\partial y}{\partial t}\right)^2+\frac{I_p}{A_c}\left(\frac{\partial\theta}{\partial t}\right)^2\right]ds \qquad \ldots(8)$$

in which ρ=mass density of material of girder, A_c=the area of cross-section, EI=flexural rigidity of girder section, GI_p=torsional rigidity of girder section, y=vertical deflection, θ=twist, L=Curved length of girder. For the girder fixed at ends the following boundary conditions are satisfied.

$$y=\frac{\partial y}{\partial s}=\theta=0 \text{ for } s=0 \text{ and } s=L \qquad \ldots(9)$$

The following mode shape function are assumed in terms of arbitrary constants A and B,

$$y=A\left[1-\cos\frac{2\pi s}{L}\right]\text{Sin } pt$$

$$\theta=B\text{ Sin}\frac{\pi s}{L}\text{ Sin } pt \qquad \ldots(10)$$

By inserting Eq. 10 into Eq. 8 and equating $V_{max}=T_{max}$, the following equation is obtained,

$$EI\left[2B^2\alpha^2L^2+8A^2\pi^4+\tfrac{32}{3}AB\pi\alpha L\right]+GI_p\left[\frac{B^2\pi^2L^2}{2}+8A^2\alpha^2\pi^2+\tfrac{32}{3}AB\pi\alpha L\right]$$
$$=\frac{\Phi^2 A_c}{2}\left\{\left(3A^2+\frac{I_p}{A_c}B^2\right)\right\} \qquad \ldots(11)$$

in which $\Phi^2=\rho p^2 L^4$. By expressing that : $\frac{\partial\Phi^2}{\partial A}=\frac{\partial\Phi^2}{\partial B}=0$, the following determinant is obtained:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \ldots(12)$$

in which,

$$a_{11}=\tfrac{3}{2}A_c\Phi^2-8\pi^2(\pi^2EI+\alpha^2GI_p)$$

$$a_{12}=a_{21}=-\frac{16}{3}\pi\alpha L(EI+GI_p)$$

$$a_{22}=\frac{I_p}{2}\Phi^2-L^2\left(2\alpha^2EI+\frac{\pi^2GI_p}{2}\right) \qquad \ldots(13)$$

The zeros of determinant will give the frequency of the curved girder.

EXAMPLES OF CURVED GIRDER

The following three problems are solved in order to demonstrate the use of numerical method presented herein. In these problems: $E=2\times10^6$ t/m², $G=1\times10^6$ t/m² and Unit weight=2.4 t/m³.

Problem 1.

Shape of girder	Circular
Cross section	Rectangular, width=0. 5m, depth=1.0 m
Span	30 m
Radius	18.7 m
Moment of Inertia	0.0417 m⁴
Torsional constant	0.052 m⁴

Problem 2.

Shape of girder	Parabolic
Cross section	Circular, diameter=1.5 m

Span	30 m
Rise	3 m
Moment of inertia	0.252 m^4
Torsional constant	0.126 m^4

Problem 3.

Shape of girder	Parabolic
Cross section	Rectangular, width=0.5 m depth=1.0 m
Span	30 m
Rise	3 m
Moment of inertia	0.0417 m^4
Torsional constant	0.052 m^4

RESULTS OF ANALYSIS

Natural frequencies and mode shapes for the three problems are obtained for four, six and eight lumped masses. The natural frequencies are tabulated in Tables 1, 2 and 3. The mode shapes for the vertical deflection and twist for eight lumped masses are shown in Figures 3 to 5. The following significant results are established from the study of the above problems:—

1. From Tables 1, 2 and 3 it is seen that frequencies of first and second mode of vibration do not differ much for different number of masses, that is, first two frequencies can be determined from an eight mass lumped system of curved girder.
2. It is seen that Rayleigh frequency based on static deflections of lumped mass system is in close agreement with the first frequency obtained from flexibility coefficient (Tables 1, 2 and 3).
3. The frequency for a uniform circular girder using Rayleigh–Ritz method is in agreement with the frequency obtained from flexibility coefficient method within 3%.
4. The frequencies of curved girder in Tables 1, 2 and 3 are also compared with those of equivalent fixed beams having span equal to curved length of beam. These results do not agree with those obtained from flexibility matrix and the frequencies of curved beam are much less than those of fixed straight beams.
5. From the mode shapes of circular and parabolic girder, Figures 3 to 5, it is seen that first mode is symmetrical and second antisymmetrical.

TABLE NO. 1

Frequencies in R/S for Circular Curved Girder of Rectangular Cross-Section

Number of masses	Frequency from influence coefficient method			Rayleigh's frequency based on Static deflection	Frequency for Equivalent fixed Beam		Frequency based on Rayleigh-Ritz Method
	First Mode	Second Mode	Third Mode		First	Second	
4	8.00	9.61	—	8.05			
6	7.60	8.99	—	7.65			
8	7.49	8.82	48.71	7.52	15.70	43.4	7.70

TABLE NO. 2

Frequencies in R/S for Parabolic Curved Girder of Circular Cross-section

Number of masses	Frequency from influence coefficient method			Rayleigh's frequency based on Static Deflection	Frequency for Equivalent fixed beam	
	First Mode	Second Mode	Third Mode		First Mode	Second Mode
4	8.72	10.25	—	8.81		
6	8.31	9.65	—	8.37		
8	8.24	9.53	60.09	8.29	9.77	26.8

TABLE NO. 3

Frequencies in R/S for Parabolic Curved Girder of Rectangular Cross-section

Number of masses	Frequency from influence coefficient method			Rayleigh's frequency based on Static Deflection	Frequency for Equivalent fixed Beam	
	First Mode	Second Mode	Third Mode		First Mode	Second Mode
4	10.51	12.44	—	10.62		
6	10.04	11.72	—	10.15		
8	9.89	11.47	70.48	9.96	19.1	52.6

CONCLUSIONS

From the study presented here the following conclusions are derived:

1. A numerical method is presented using flexibility coefficients for obtaining natural frequencies and modes of a symmetrical fixed single span curved girder of arbitrary shape and arbitrary stiffness considering bending and torsion. Any number of modes can be determined from this method.
2. The numerical method is used to obtain frequencies of circular and parabolic girders. The fundamental frequency of circular girder has close agreement with that obtained from Rayleigh—Ritz procedure.
3. First two modes of vibration can be obtained from an eight mass lumped system. Higher modes can be determined using more number of masses.

ACKNOWLEDGEMENT

Authors gratefully acknowledge the valuable suggestions that they received from Dr. A. S. Arya, Professor and Head, School of Research and Training in Earthquake Engineering, Roorkee in the preparation of this paper.

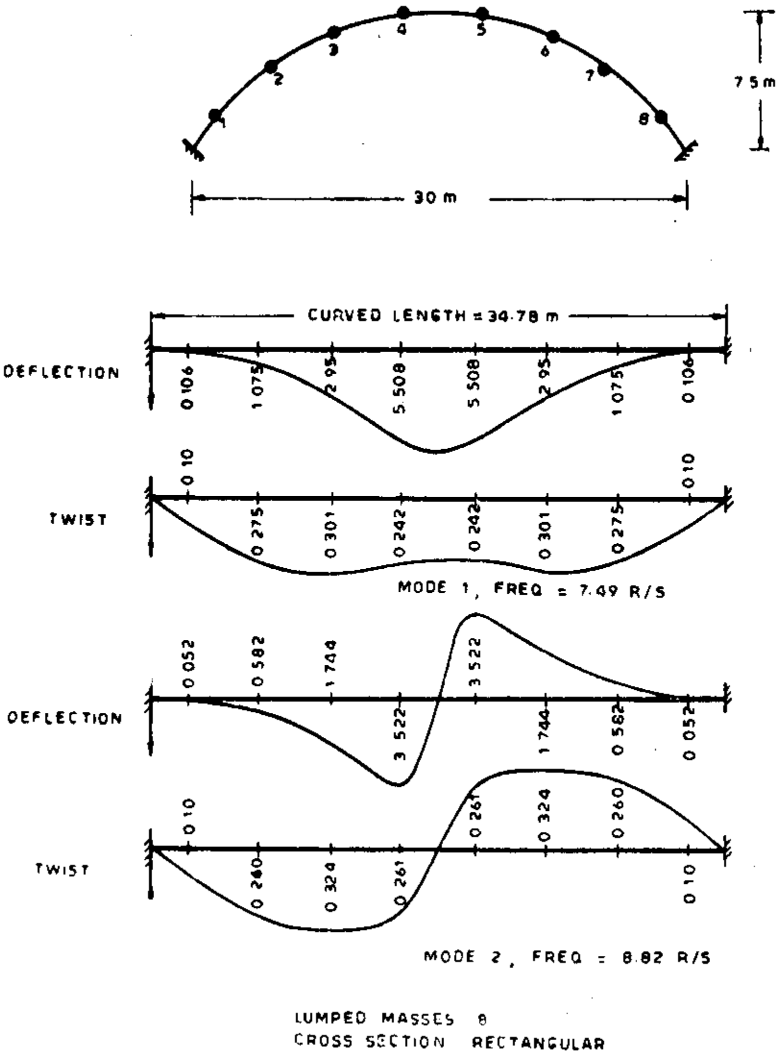

Fig. 3. Mode shapes of circular curved girder

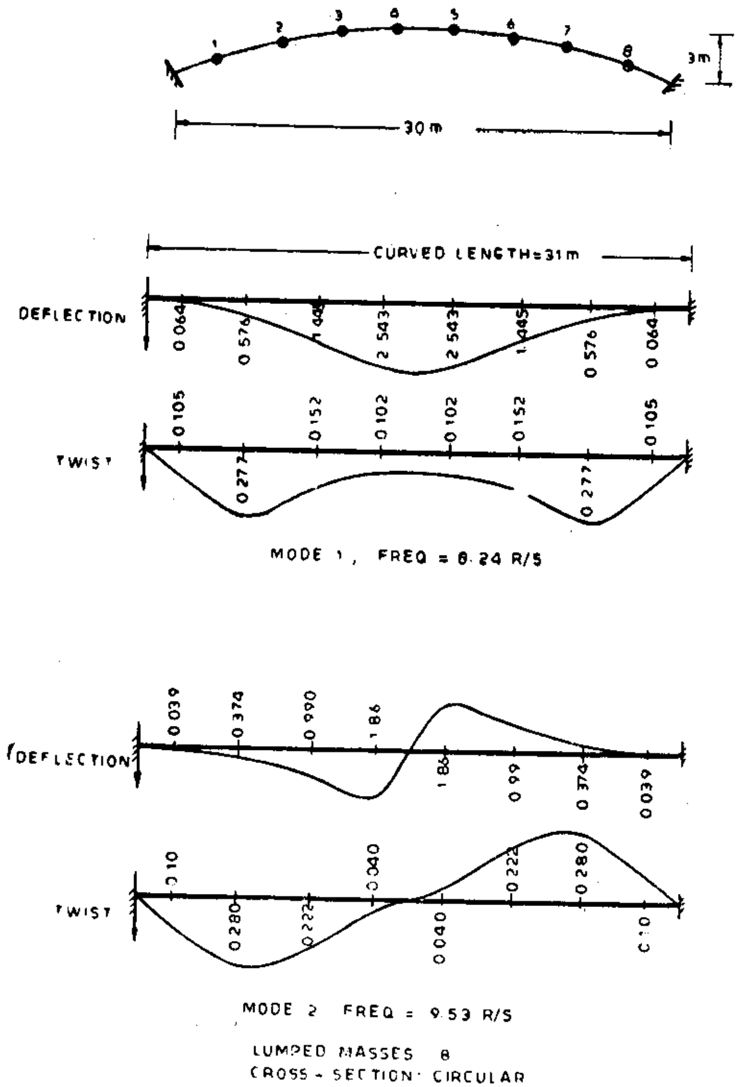

Fig. 4. Mode shapes of parabolic curved girder

REFERENCES

1. Culver, C. G., 'Natural frequencies of horizontally curved beams', Proceedings of the ASCE, Journal of the Structural Division, Vol. 93, No. ST2, April 1967.
2. Qamaruddin, M., 'Dynamic Response of Horizontally Curved girders', M. E. Thesis, University of Roorkee, Roorkee, November 1969.
3. Reddy, M. N., 'Lateral Vibrations of plane curved bars', Proceedings of the ASCE, Journal of the Structural Division, Vol. 94, No. ST10, October, 1968.

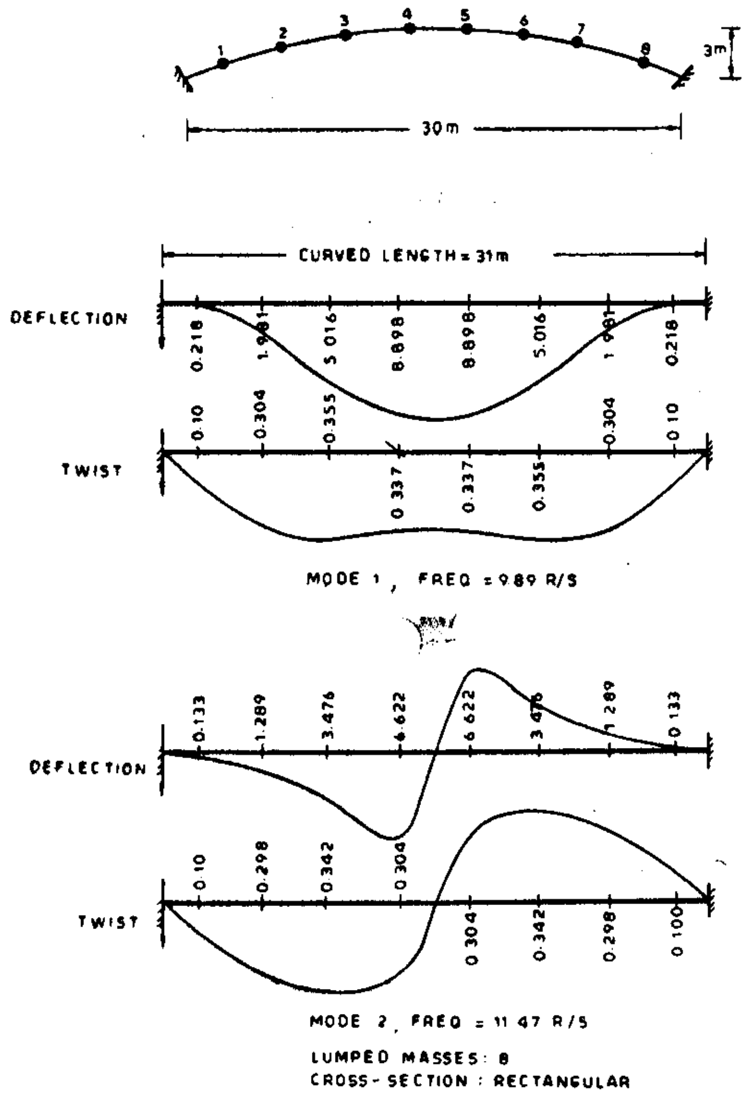

Fig. 5. Mode shapes of parabolic curved girder

4. Tan, C. P. and Shore, 'Dynamic response of a horizontally curved bridge', Proceedings of the ASCE, Journal of the Structural Division, Vol. 94, No. ST3, March 1968.
5. Tan, C. P. and Shore, 'Dynamic response of a horizontally curved girder bridge to moving load', Proceedings of the ASCE, Journal of the Structural Division, Vol. 94, No. ST9, September 1968.
6. Volterra, E. and Morell, J. D., 'Lowest natural frequency of elastic arc for vibrations outside plane of initial curvature', Transactions of the ASME, V-28, series E No. 4, December 1961.