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ANALYTICAL PROSPECTS IN NONLINEAR STRUCTURAL DYNAMICS

Gurmail S. Benipal

Department of Civil Engineering
Indian Institute of Technology Delhi
Hauz Khas, New Delhi-110016
E-mail: *gurmail@civil.iitd.ac.in*

ABSTRACT

Currently, the dynamic response of nonlinear structures is generally determined by numerical integration of the incremental coupled second order differential equations of motion involving their tangent stiffness matrices. However, the dominant paradigm being computational, the initial value problems (IVPs) being solved are rarely stated in an analytical form. Three new formulations of analytical IVPs for nonlinear structures are presented in this Paper. Firstly, it is argued that only the IVPs based upon coupled third order differential equations of motion are proper for nonlinear MDOF dynamical systems. Secondly, it is shown that, for homogeneous dynamical systems, coupled second order differential equations are applicable. Finally, for general nonlinear dynamical systems, it is proposed to formulate the IVPs using decoupled nonlinear second order differential equations. Theoretical significance of these three analytical prospects in nonlinear structural dynamics is discussed.

KEYWORDS: Nonlinear Structural Dynamics, Third-Order Equations of Motion, Homogeneous Systems, Decoupled MDOF Systems

INTRODUCTION

As in rest of classical mechanics, Newton's second law of motion is the basis of structural dynamics. Starting from their basic formulation in the year 1877 by Rayleigh [1], the matrix form of equations of motion for linear MDOF structures was proposed by Collar, Duncan and Frazer in the third decade of twentieth century [2]. The following second-order coupled linear differential equations of motion for vibrating MDOF linear elastic structures with nodal masses and forces are popular till now in civil, mechanical and aeronautical engineering disciplines:

$$M\ddot{u} + C\dot{u} + Ku = F(t) \tag{1}$$

Here, the symbols M, C and K represent the constant mass, damping and stiffness matrices of the structure while $u(t)$ represent its instantaneous nodal displacement response to the applied forcing function $F(t)$. As usual, superposed dot implies time derivative. For complete formulation of initial value problem (IVP), the initial nodal displacements $u(0)$ and velocities $\dot{u}(0)$ are required to be specified. Under the assumption of classical damping, the damping matrix of these structures is generally formulated in terms of their mass and stiffness matrices. Linear mode theory valid for these structures is well-developed [3, 4, 5].

Nonlinear dynamical systems theory having applications across the disciplines deals primarily with SDOF systems. Duffing oscillators are known to exhibit sub-harmonic resonances, extreme sensitivity to initial conditions, exciting forces and system parameters, chaotic motions, etc. Self-excited vibrations are predicted for van der Pol oscillators. Analytical solutions being rarely possible, the nonlinear dynamic response is generally predicted by numerical integration of the equations of motion [6].

The theory of MDOF nonlinear dynamical systems is not so well developed. According to Lyapunov, for an n-DOF conservative system without internal resonances, there exist at least n periodic solutions about the equilibrium state. These stable periodic motions constitute the normal modes for dynamical systems. Analogous to normal modes in linear dynamical systems, these invariant periodic motions are called nonlinear normal modes for nonlinear systems, even though these modes lack orthogonality and violate the principle of superposition [7, 8]. However, it has been recognised [9] that these nonlinear modes vary with instantaneous total amplitude of vibration. Also, in view of the pure uncoupling of linear modes, the concept of 'mode interactions' is considered by some researchers to be an oxymoron [10].

A general computational strategy for determining the dynamic response of nonlinear MDOF structures formulated by Wilson, Farhoomand and Bathe [11] is very popular. The incremental equation of motion is deduced as follows from the incremental balance of elastic, damping, inertial and applied forces denoted as $F_e(t),\ F_d(t),\ F_i(t)$ and $F(t)$ respectively. At any instant during the motion, the elastic, damping and inertial forces are defined by the constitutive functions of instantaneous elastic displacements, velocities and accelerations respectively as

$$F_e(t) = F_e(u(t)),\ F_d(t) = F_d\big(\dot{u}(t)\big)\ and\ F_i(t) = F_i(\ddot{u}(t)) \qquad (2)$$

The increments in these forces are determined by the increments in the corresponding kinematic variables in incremental duration Δt as below:

$$\Delta F_e(t) = K^t \Delta u(t),\ \Delta F_d(t) = C^t \Delta \dot{u}(t)\ and\ \Delta F_i(t) = M \Delta \ddot{u}(t) \qquad (3)$$

Here, K^t represents the tangent stiffness matrix defined as $K_{ij}^t = \partial F_{ei}/\partial u_j$. Similarly, the tangent damping matrix, C^t, is defined as $C_{ij}^t = \partial F_{di}/\partial \dot{u}_j$. However, the mass matrix M remains constant irrespective of the instantaneous acceleration. The incremental balance of these forces ($\Delta F_i(t) + \Delta F_d(t) + \Delta F_e(t) = \Delta F(t)$) yields the following incremental form of governing equation of motion:

$$M \Delta \ddot{u} + C^t \Delta \dot{u} + K^t \Delta u = \Delta F(t) \qquad (4)$$

Like the tangent stiffness matrix is defined in terms of instantaneous displacements, the tangent damping matrix of the nonlinear structures has to be established in terms of instantaneous velocities using methods of system identification. In contrast, as per the practice in the linear structural dynamics, the damping matrix of nonlinear structures is specified independently or in terms of instantaneous tangent stiffness matrix and constant mass matrix under assumption of instantaneous classical damping. In this latter case, both the tangent stiffness and damping matrices are determined by the instantaneous displacements. Sometimes, assuming the nonlinear structure to be initially classical damped, the damping matrix is determined from the initial tangent stiffness matrix and thereafter assumed to remain constant. This incremental form of equation of motion is particularly useful for determination of structural response by numerical integration at discrete time intervals. The popular numerical integration schemes include Newmark-beta method, Wilson-theta method, Runge-Kutta method, etc. Though not explicitly mentioned, this numerically-formulated IVP also requires specifications of initial nodal displacements and velocities [3, 4, 5, 12, 13].

It must be kept in mind that, in classical mechanics, the problems are first formulated in the analytical form and then attempts are made to construct their closed-form solutions. It is possible to do so only for quite simple problems. Numerical methods and computer software need to be employed for solving more complex problems. Since most of the problems of engineering importance are complex, a distinct discipline of computational mechanics has emerged. However, one cannot start with the numerical version of the problem as has been done in the case of nonlinear structural dynamics. It is only by the formulation of analytical IVPs that it is possible to identify the algebraic structure informing the structural response. Then and only then efficient analytical and computational methods of their solution can be formulated and new applicable theorems and principles can be proved.

The objective of the present Paper is to suggest new analytical prospects for moving ahead in nonlinear structural dynamics. Firstly, a new IVP involving coupled third order nonlinear differential equations of motion is formulated. Secondly, it is shown that the classical formulation of IVPs based upon coupled second order differential equations is valid for a class of nonlinear structures --- homogeneous dynamical structures --- as well. Thirdly, an IVP involving decoupled second-order differential equations is proposed for general nonlinear MDOF structures. Lastly, the proposed analytical approach to nonlinear structural dynamics is critically evaluated for its role in further developments in this field.

GENERAL NONLINEAR DYNAMICAL SYSTEMS

For an SDOF nonlinear elastic structure, the nonlinear force-displacement relation, $P(u)$, can always be stated in terms of its secant stiffness (K^s) as $P(u) = K^s u$. It is not possible to formulate unique secant stiffness matrices for the nonlinear elastic MDOF structures. For example, the following constitutive equations can be deduced for the nonlinear elastic system proposed by Qaisi and Kilani [14]:

$$\begin{aligned} P_1 &= Au_1 + Bu_2 + Cu_1^3 + Du_1^2u_2 + Eu_1u_2^2 + Fu_2^3 \\ P_2 &= Gu_1 + Hu_2 + Iu_1^3 + Ju_1^2u_2 + Ku_1u_2^2 + Lu_2^3 \end{aligned} \qquad (5)$$

The above constitutive equation can be restated using any one of the following two versions of the secant stiffness matrix:

$$K_{11}^{s} = A + Cu_1^2 + Du_1u_2 + Eu_2^2\,, \qquad K_{12}^{s} = B + Fu_2^2$$
$$K_{21}^{s} = G + Iu_1^2 + Ju_1u_2 + Ku_2^2\,, \qquad K_{22}^{s} = H + Lu_2^2 \tag{6a}$$
$$K_{11}^{s} = A + Cu_1^2, \qquad K_{12}^{s} = B + Du_1^2 + Eu_1u_2 + Fu_2^2$$
$$K_{21}^{s} = G + Iu_1^2\,, \qquad K_{22}^{s} = H + Ju_1^2 + Ku_1u_2 + Lu_2^2 \tag{6b}$$

This conclusion applies even to the case of geometrically nonlinear structures wherein the secant stiffness matrix is composed of elastic and geometric stiffness matrices [15].

Lack of unique secant stiffness matrices for nonlinear structures implies that it is not possible to state the analytical second order matrix differential equation of motion for them as (Equation 1) for linear structures. However, their tangent stiffness matrix determined as $K_{ij}^{t} = \partial P_i/\partial u_j$ is always unique. Thus, it is possible to state their constitutive equations in the rate-form as $\dot{P}_i = K_{ij}^{t}\,\dot{u}_j$ or in the index-free notation as $\dot{P} = K^t\dot{u}$. For undamped structures, the instantaneous internal force vector is obtained as $P(t) = F(t) - M\ddot{u}$. On differentiating both sides, one gets $\dot{P} = \dot{F} - M\dddot{u}$. In view of this, the following coupled third order nonlinear differential equation of motion is derived for undamped MDOF nonlinear dynamical systems:

$$M\dddot{u} + K^t\dot{u} = \dot{F} \tag{7}$$

The above equation of motion for undamped systems is generalised to obtain the governing equation of motion for damped systems as

$$M\dddot{u} + C\ddot{u} + K^t\dot{u} = \dot{F} \tag{8}$$

This IVP involving third-order nonlinear differential equations requires the specification of initial nodal displacement, velocity and acceleration vectors. However, as argued earlier for sagging elastic cables by the Author and co-workers [16], initial acceleration vector is determined from independently specified initial nodal displacement and velocity vectors using Newton's law of motion as follows:

$$\ddot{u}(0) = M^{-1}[F(0) - P(0) - C\dot{u}(0)] \tag{9}$$

Here, the initial force vector $P(0)$ corresponds to the initial displacement vector.

The above third-order nonlinear differential equations of motion with nodal displacements as primary variables can also be expressed as the following equivalent second order nonlinear differential equations of motion with nodal velocities ($\dot{u} = V$) as primary kinematic variables:

$$M\ddot{V} + C\dot{V} + KV = G(t) \tag{10}$$

Here, $G(t) = \dot{F}(t)$ represents the forcing function. This version of equation of motion is particularly amenable to application of available numerical integration techniques.

HOMOGENEOUS DYNAMICAL SYSTEMS

A function $f(X_1, X_2, -, -, X_n)$ is said to be function homogeneous of order m of n independent variables (X_i), if it can be expressed as

$$f(X_1, X_2, -, -, X_n) = X_r^m\, g({}^{X_1}/_{X_r}, {}^{X_2}/_{X_r}, -, -, {}^{X_n}/_{X_r}) \tag{11}$$

Here, the function $g(X_1, X_2, -, -, X_n)$ is zero order homogeneous function of these variables. Euler's theorem for homogeneous functions is stated below for ready reference [17]:

$$\left(\frac{\partial f}{\partial X_i}\right) X_i = m\, f(X_1, X_2, -, -, X_n) \tag{12}$$

As per the time-honoured taxonomical practice in structural theory, the structures are classified as linear or nonlinear according to the mathematical function relating loads and displacements. Thus, a structure is said to be an m-order homogeneous mechanical system if the nodal forces are functions homogeneous of order m of nodal displacements [16, 19]. In view of Euler's theorem,

$$\left(\frac{\partial P_i}{\partial u_j}\right) u_j = m\, P_i(u_1, u_2, -, -, u_n) \tag{13}$$

Or equivalently,

$$\left(K_{ij}^{t}\right) u_j = m\, P_i(u_1, u_2, -, -, u_n) \tag{14}$$

In the index-free notation, one gets $\frac{1}{m} K^t u = P$. The total nodal forces and displacements are related through secant stiffness matrix K^s as in $K^s u = P$ where $K^s = \frac{1}{m} K^t$. For example, the particular nonlinear constitutive equations (Equation 5) can also be restated as

$$P_1 = Q_1 + R_1, \qquad Q_1 = Au_1 + Bu_2, \qquad R_1 = Cu_1^3 + Du_1^2 u_2 + Eu_1 u_2^2 + Fu_2^3$$
$$P_2 = Q_2 + R_2, \qquad Q_2 = Gu_1 + Hu_2, \qquad R_2 = Iu_1^3 + Ju_1^2 u_2 + Ku_1 u_2^2 + Lu_2^3 \tag{15}$$

It can be observed that the force vector components, Q_i and R_i, respectively are first and third order homogeneous functions of displacements. The tangent and secant stiffness matrices (K^t and K^s) of this homogeneous mechanical system can be established as follows:

$$\frac{\partial P_i}{\partial u_j} = \frac{\partial Q_i}{\partial u_j} + \frac{\partial R_i}{\partial u_j}, \qquad K^t = K_Q^t + K_R^t, \qquad K^s = K_Q^t + \frac{1}{3} K_R^t \tag{16}$$

Some other researchers have investigated planar dynamical system with the following fourth degree homogeneous polynomial potential [18]:

$$V(u_1, u_2) = au_1^4 + bu_1^3 u_2 + cu_1^2 u_2^2 + du_1 u_2^3 + eu_2^4 \tag{17}$$

To recapitulate, the strain energy function for conventional two-DOF linear elastic structures is stated as follows:

$$V(u_1, u_2) = au_1^2 + bu_1 u_2 + cu_2^2 \tag{18}$$

Of course, it is a second degree homogeneous polynomial potential. More general second order homogeneous complementary energy potential has been derived for a particular two-DOF cracked concrete beams by Pandey and Benipal [19]. Such second order homogeneous complementary energy potentials assume the form of polynomial fractions as

$$\Omega(P_1, P_2) = S_1(P_1, P_2) / S_2(P_1, P_2)$$
$$S_1(P_1, P_2) = AP_1^3 + BP_1^2 P_2 + CP_1 P_2^2 + DP_2^3, \qquad S_2(P_1, P_2) = P_1 + EP_2 \tag{19}$$

Using this expression for the complementary energy potential, the force-displacement relations can be derived using Castigliano Theorem. Their displacement vector components turn out to be functions homogeneous of order unity of the force vector components. This is why such structures are said to belong to the class of First Order Homogeneous Mechanical (FOHM) Systems. The linear systems do belong, albeit merely vacuously, to the same class of FOHM systems. The secant flexibility (stiffness) matrices of these mechanical systems equal their tangent flexibility (stiffness) matrices. Similarly, the Falconi-Lacomba-Vidal potential (Equation 17) [18] defines a Third Order Homogeneous Mechanical System such that its secant stiffness matrix equal one-third of its tangent stiffness matrix.

Thus, the unique secant stiffness matrices of the homogeneous dynamical systems can be obtained from their tangent stiffness matrices. The following coupled second order nonlinear differential equations of motion stated in terms of such secant stiffness matrices can be used for predicting their nonlinear dynamic response:

$$M\ddot{u} + C^s \dot{u} + K^s u = F(t) \tag{20}$$

Here, the damping matrix, when established from the instantaneous secant stiffness matrix, depends upon the instantaneous displacements. Of course, the instantaneous damping forces are linear functions of the instantaneous velocities. Alternatively, the initial damping matrix can be assumed to remain constant. In this latter case, the structures will be rendered nonclassically damped resulting in gradually varying mode shapes.

Sagging cables are known to lack unique passive natural state configuration. However, corresponding to each equilibrium state, there exists a unique natural state in reference to which their elastic displacements are defined. It has been established by Kumar, Ganguli and Benipal [16] that the natural state coordinates (x_i) and small elastic displacements (u_i) of MDOF weightless sagging elastic cables are functions homogeneous of order zero and unity respectively of the applied nodal forces (P_r). The deformed state nodal coordinates or placements $(y_i = x_i + u_i)$ are determined uniquely by these applied forces. The tangent configurational, elastic and elasto-configurational flexibility matrices are defined respectively as $D_{ij} = \partial x_i / \partial P_j$, $f_{ij} = \partial u_i / \partial P_j$ and $N_{ij} = D_{ij} + f_{ij} = \partial y_i / \partial P_j$. Although sagging elastic cables belong to the class of Homogeneous Mechanical Systems, it is not possible to even define the constitutive equation of the type $y = L^s P$ in terms of some secant flexibility matrix L^s. This is because of the fact that, in the case of elasto-flexible sagging cables, the kinematic vector (y) represents deformed state nodal placements, not the elastic nodal displacements (u) as in the case of common elastic structures. Of course, the following rate-type constitutive equations can be stated for these structures: $\dot{y} = N\dot{P}$ or $\dot{P} = K^t\dot{y}$ where tangent stiffness matrix $K^t = N^{-1}$. As argued above for General Dynamical Systems, following third order nonlinear differential equation of motion is appropriate for damped weightless MDOF sagging elastic cables:

$$M\dddot{y} + C\ddot{y} + K^t\dot{y} = \dot{F} \qquad (21)$$

Application of Newton's second law of motion to undamped weightless MDOF sagging elastic cables yields the equation of dynamic equilibrium as $F(t) - P(t) = M\ddot{y}$. It can be observed that it is the absolute acceleration $(\ddot{y} = \ddot{x} + \ddot{u})$ defined in reference to the absolute space which appears in the equation of motion. As sagging cables lack unique natural state, the elastic displacements (u) are measured relative to their instantaneous natural state (x). Thus, it is not the elastic acceleration $(\ddot{u})$, but total elasto-configurational acceleration $(\ddot{y})$ which represents the Newtonian absolute acceleration. In contrast, the elastic displacements (u) of the conventional elastic structures are measured from their unique natural state (x). For such structures, the elastic acceleration $(\ddot{u})$ does appear in their equations of motion.

DECOUPLED NONLINEAR DYNAMICAL SYSTEMS

To recapitulate, the constitutive equations of nonlinear elastic systems are stated in terms of nonlinear force-displacement relations. Force vector components are expressed as nonlinear functions of the displacement vector components. These nonlinear force-displacement relations can also be expressed in the following 'uncoupled' form:

$$P_i = K_i^s(u_1, u_2, -, -, u_n)\, u_i \qquad (22)$$

Here, the functions $K_i^s(u_1, u_2, -, -, u_n)$ represent the secant stiffness coefficients relating force vector components (P_i) with the corresponding displacement vector components (u_i). However, these secant stiffness coefficients (K_i^s) determined by the instantaneous value of displacement vector components do not remain constant during vibration. Using their force-displacement relations in this form, an n-DOF nonlinear structure can be modelled as n uncoupled SDOF nonlinear structures resembling Duffing oscillators. The governing second order nonlinear differential equations of motion for each of these SDOF systems can be stated as

$$M_i\ddot{u}_i + C_i\dot{u}_i + K_i^s u_i = F_i(t) \qquad (23)$$

To an extent, such decoupling is routinely employed in the structural stability theory. While investigating the elastic stability of beam-columns under the action of lateral and axial loads, it is common practice to state the secant lateral load-displacement relations wherein the stiffness coefficient depends upon the axial force [15]. Thus, a two-DOF system is converted into an SDOF system.

It is well known that the stiffness coefficient of a genuine SDOF stable structure is always positive as a non-positive stiffness coefficient would imply lack of stability. In contrast, the stiffness coefficient of the equivalent SDOF structures derived from an nonlinear MDOF structure can assume all real values including positive, nil and negative values. However, in this case, vanishing or negative stiffness coefficients of SDOF structures do not imply their instability. Of course, an underdamped nonlinear SDOF structure with positive instantaneous stiffness exhibits oscillatory motion. In contrast, even undamped SDOF structures with vanishing or negative stiffness are expected to experience only non-oscillatory motion. Damping coefficients for SDOF structures with positive instantaneous stiffness can be assigned after due system identification. Though, system identification of nonlinear structures is quite complex, assignment of damping coefficients does not pose any theoretical challenge. When the stiffness is negative, critical damping coefficient $(C_{icr} = 2\sqrt{K_i^s M})$ turns out to be imaginary. It is not yet clear how to assign the values of damping ratio and damping coefficient.

It turns out that nonlinear seismic analysis of structures using such second order decoupled equations of motion has recently been attempted. The conventional secant stiffness matrix is diagonalized by using 'equivalent nodal secant matrix' [20]. The instability caused by nonpositive secant stiffness coefficients as identified above has not been encountered. Perhaps, this is because of the fortuitous choice of the particular structure and loading resulting in positive stiffness matrix. It is only the secant stiffness matrices (K_{ij}^s) with some negative off-diagonal coefficients that the corresponding secant stiffness coefficients (K_i^s) are expected to vanish or even be negative.

DISCUSSION

It should be appreciated that different classes of mechanical systems are identified by their constitutive equations. However, constitutive equations of structures as distinct from their motion are rarely mentioned in the discourse on structural theory. The current philosophy of seismic design of structures relies on their ductility for the purpose of seismic energy dissipation. As per the professional parlance in the earthquake engineering community, the term 'nonlinear analysis' is deemed to convey inelastic analysis [4, 13]. It should be kept in mind that inelastic structures are always nonlinear, though the elastic structures can either be linear or nonlinear. Thus, the class of physically nonlinear elastic structures under discussion here as well as in nonlinear dynamical systems theory is not even considered admissible! Of course, energy dissipation by damping, though not by elastoplastic flow, is incorporated. Examples of distinct classes of structures include the linear elastic conservative structures, Kelvin-Voigt dynamical systems, classically damped systems, etc. Class of homogeneous dynamical systems has only recently been introduced: Cracked concrete beams and weightless sagging cables undergoing small elastic displacements have been modelled respectively as first order and mixed order homogeneous dynamical systems by the Author and co-workers [16, 19]. A new class of decoupled nonlinear MDOF dynamical systems is explicitly identified for the first time in this Paper.

Even in the theories of vibrations of damped SDOF structures proposed since Rayleigh [1], it is the Kelvin-Voigt rheological model which is generally employed for deriving the governing second order linear differential equations of motion. The Zener rheological model is claimed by Haan and Sluimer to be better suited for predicting viscoelastic response of concrete. However, it demands third order differential equations of motion for predicting dynamic response of concrete structures [21]. Third order differential equations of motion have also been employed to describe the dynamic response of such mechanical systems like elastically-coupled isolator systems, relaxation type of vehicle suspension, vehicle-bridge interaction and nonlinear control systems [22,23], jerk phenomenon [24] and muscle-tendon systems [25]. Earlier research investigations into the field of third order oscillators were carried out by Rauch, Friedrichs and Sherman [25].

As stated earlier, the IVP corresponding to the incremental equation of motion (Equation 4) is not formulated explicitly, though it is implied that only initial displacements and velocities need to be specified. Analytical form of the equation of motion (Equation 8) can be deduced from the incremental equation of motion (Equation 4) by dividing both sides by duration of the time interval Δt. Then, using the relations ($\Delta\ddot{u} = \dddot{u}\Delta t,\ \Delta\dot{u} = \ddot{u}\Delta t,\ \Delta u = \dot{u}\Delta t,\ \Delta F(t) = \dot{F}\Delta t$), one obtains the third order differential equation of motion (Equation 8). However, the third order differential equation of motion demands the specification of initial displacements, velocities and accelerations. As presented in this Paper, such equation of motion along with these initial conditions leads to explicit formulation of IVP valid for MDOF nonlinear structures.

Linear modal analysis results in n uncoupled third-order homogeneous ordinary differential equations. The auxiliary equation for each of these differential equations with real coefficients is cubic polynomial with at least one real negative/positive root. The other two roots can be real distinct, real equal or complex conjugate depends upon the numerical values of its real coefficients [26]. For underdamped vibration modes, these latter roots are complex conjugate. Such systems with one real negative root and two complex conjugate roots are shown to exhibit their characteristic mode of damping of free vibrations. This involves exponential decay of oscillatory vibrations without change of sense of vibration amplitude [21]. As expected, such characteristic damping mode has also been predicted for weightless sagging elastic planar 4-DOF cables obeying third order differential equations of motion [16].

No attempt has been made here to investigate the problem of equivalence between the popular incremental and the proposed analytical IVPs for nonlinear dynamical systems. However, it is clear that their respective equations of motion, (Equation 4) and (Equation 8), are equivalent, but their required initial conditions do differ. More significantly, the IVP based upon incremental second order differential equation of motion is oblivious of the presence of a mode of damping in addition to the well-known underdamped and over damped free motion. This mode arises naturally when the third order differential equation of motion is employed.

A comparison of equations of motion (Equation 1 and Equation 10) reveals the similarity of their mathematical form. It can be observed that displacements and velocities are the primary kinematic variables respectively for the linear and nonlinear dynamical systems. Mathematical similarity ensures same modal frequencies ($M^{-1}K$) for both the types of systems. However, modal vectors are displacement vectors for linear systems, while it is velocity vectors which represent the modal vectors for nonlinear systems. For the particular case of forcing function ($F(t) = F_0 + F_t \, sin\omega t$) where F_0 and F_t represent sustained constant and small peak harmonic force vectors respectively, the resonance is predicted in both the formulations to occur at the same modal frequencies. Particular vibration mode can be excited by selecting the additional peak harmonic force vector F_t proportional to the corresponding modal vector.

For vibrating nonlinear systems, tangent stiffness matrices are determined by the instantaneous displacements. Thus, the instantaneous modal frequencies determined by the instantaneous tangent stiffness matrices vary with time-dependent instantaneous displacements. For the forcing function ($F(t) = F_0 + F_t sin\omega t$), the instantaneous modal frequencies do not differ very much from those determined using the tangent stiffness matrix in the equilibrium state provided the displacements from the equilibrium state remain small. For the extreme case of linear dynamical systems, the tangent and secant stiffness matrices, being equal and constant, are independent of the displacements. For homogeneous dynamical systems and decoupled MDOF nonlinear dynamical systems, the secant stiffness matrices have been defined which are employed for stating their second order differential equations of motion. It should be remembered that still, it is only the instantaneous tangent, not secant, stiffness matrices which determine the instantaneous modal frequencies. Case of first order homogeneous dynamical system is different in that, their instantaneous secant and tangent stiffness matrices being equal, their secant stiffness matrix can be employed for instantaneous modal analysis. Further, cumbersome modal analysis is not required for n SDOF decoupled dynamical systems equivalent to MDOF nonlinear dynamical systems.

As argued so forcefully by Truesdell [27], in physical sciences including structural dynamics, both computation and experimentation should properly by conducted under the supervision of a well-developed mathematical theory. Predictions made by using 'floating' computational methods for all types of nonlinear systems can be misleading. Contrary to this understanding, the commercial computational algorithms have taken precedence over the analytical formulation of the IVPs in nonlinear dynamics of MDOF structures. In this Paper, an attempt has been made to reintroduce the analytical paradigm in this field.

CONCLUSIONS

In this Paper, the analytical IVP involving third order nonlinear differential equations of motion appropriate for nonlinear MDOF structures is proposed. For a particular class of MDOF nonlinear structures, i. e., homogeneous mechanical systems, the conventional IVP based upon second order nonlinear differential equations of motion is shown to be valid. Also, an entirely different approach based upon decoupled MDOF nonlinear mechanical systems is proposed. Going against the dominant computational paradigm, it is argued that it is only after the formulation of analytical IVPs that new theorems and principles can be proved and dedicated analytical and computational solution methods can be formulated. Possible complete equivalence between the numerical and analytical IVPs based respectively upon second order incremental and third order analytical differential equations of motion has yet to be established. In the case of homogeneous dynamical systems, some initial progress has been made by researchers including the present Author. For the decoupled version of the nonlinear MDOF structures, even the formulation of analytical IVP has barely been suggested in this Paper, though such an approach has been adopted by others for numerical prediction of seismic response of some structures. It is hoped that the present Paper will motivate research along these identified analytical prospects.

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