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RESIDUAL RESPONSE OF A MULTILAYERED HALF-SPACE TO TWO-DIMENSIONAL SURFACE LOADS

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ABSTRACT

The two-dimensional problem of the static deformation of a multilayered half-space by surface loads is studied in detail. Both plane strain and antiplane strain cases are considered. The Thomson-Haskell matrix method is used to obtain the field at any point of the medium. Explicit expressions for stresses caused by a surface line load on a uniform half-space are derived. The formulation developed is quite convenient for numerical computation.

KEY WORDS : Static Deformation, Multilayered Half-space, Surface Loads, Plane Strain, Antiplane Strain.

INTRODUCTION

The classical Boussinesq solution to the problem of a normal static load on the surface of a semi-infinite elastic medium offers wide applications to loading problems in geophysics and engineering. This solution has since been extended to multilayered configurations. Kuo (1969) studied the three dimensional problem of inclinded static loads on the surface of a multilayered medium. Singh (1970) used the Thomson-Haskell matrix method to study the static deformation of a multilayered half-space by three dimensional sources. Recently, Singh and Garg (1985) tackled the corresponding two-dimensional problem of a long displacement dislocation in a multilayered half-space.

In the present paper, we formulate the two-dimensional problem of the static deformation of a multilayered half-space by surface loads. Both plane strain and antiplane strain cases are considered. The Thomson-Haskell matrix method (Thomson, 1950; Haskell, 1953) is used to obtain the static field at any point of the multilayered half-space for given surface loads. The particular cases of a normal line load and a shear line load are considered in detail. It is shown that in the case of a uniform half-space the integrals giving the stresses can be integrated analytically.

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2. BASIC EQUATIONS

We shall be considering a two-dimensional approximation in which the displacement components (u_1, u_2, u_3) are independent of x_1 so that $\frac{\partial}{\partial x_1}=0$. Under this assumption, the plane strain problem $(u_1=0)$ and the antiplane strain problem $(u_2 = u_3 = 0)$ are decoupled and, therefore, can be treated separately.

In the case of the plane strain problem the equilibrium equations for zero body forces are

$$(\lambda+\mu) \frac{\partial}{\partial x_i}\left(\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right) +\mu\nabla^2 u_i = 0, \qquad (i=2, 3) \tag{2.1}$$

where λ, μ are the Lame' parameters and

$$\nabla^2 = \frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}. \tag{2.2}$$

The non-zero stresses are

$$\begin{aligned} p_{11} &= \lambda\left(\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right), \\ p_{22} &= (\lambda+2\mu)\frac{\partial u_2}{\partial x_2}+\lambda\frac{\partial u_3}{\partial x_3}, \\ p_{33} &= \lambda\frac{\partial u_2}{\partial x_2}+(\lambda+2\mu)\frac{\partial u_3}{\partial x_3}, \\ p_{23} &= \mu\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right). \end{aligned} \tag{2.3}$$

For antiplane strain problem, u_1 satisfies the equation (for zero body force)

$$\nabla^2 u_1 = 0. \tag{2.4}$$

The non-zero stresses are

$$p_{12} = \mu\frac{\partial u_1}{\partial x_2}, \quad p_{13} = \mu\frac{\partial u_1}{\partial x_3}. \tag{2.5}$$

3. MULTILAYERED HALF-SPACE

We consider a semi-infinite elastic medium made up of p-1 parallel, homogeneous, isotropic layers lying over a homogeneous, isotropic half-space. The layers are numbered serially, the topmost layer being layer 1. The half space is designated as layer p. The origin of the cartesian coordinate system (x_1, x_2, x_3) is placed at the surface with the x_3-axis drawn into the medium. The nth layer is of thickness d_n and has Lame' parameters λ_n, μ_n. It is bounded by the interfaces $z = z_{n-1}, z_n$ (we write y for x_2 and z for x_3). Evidently, $z_0 = 0$ and $z_{p-1} = H$, where H is the depth of the last interface.

3.1 Plane Strain

The plane strain problem can be solved in terms of the Airy stress function U such that

$$p_{22} = \frac{\partial^2 U}{\partial z^2}, \quad p_{23} = -\frac{\partial^2 U}{\partial y \, \partial z}, \quad p_{33} = \frac{\partial^2 U}{\partial y^2}, \tag{3.1}$$

$$\nabla^2 \nabla^2 U = 0. \tag{3.2}$$

A solution of the biharmonic equation (3.2) is of the form

$$U = \int_0^\infty (Ae^{-kz} + Be^{kz} + Ckze^{-kz} + Dkze^{kz}) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} dk, \tag{3.3}$$

where A, B, C, D may be functions of k. Following Singh and Garg (1985) we find

$$u_2 = \int_0^\infty V \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} k \, dk, \tag{3.4}$$

$$u_3 = \int_0^\infty W \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k \, dk, \tag{3.5}$$

$$p_{23} = \int_0^\infty S \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} k^2 \, dk, \tag{3.6}$$

$$p_{33} = \int_0^\infty N \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k^2 \, dk. \tag{3.7}$$

The functions V, W, S, N are given by the matrix relation

$$[A(z)] = [Z(z)] \, [K], \tag{3.8}$$

where

$$[A(z)] = [V, W, S, N]^T, \quad [K] = [A, B, C, D]^T \tag{3.9}$$

and $[\,..]^T$ denotes the transpose of the matrix $[\ldots]$. The matrix $[Z(z)]$ is given in (3.10)

$$[Z(z)] = \begin{bmatrix} \frac{-1}{2\mu} e^{-kz} & \frac{-1}{2\mu} e^{kz} & \frac{1}{2\mu}\left(\frac{1}{\alpha} - kz\right) e^{-kz} & \frac{-1}{2\mu}\left(\frac{1}{\alpha} + kz\right) e^{kz} \\ \frac{1}{2\mu} e^{-kz} & \frac{-1}{2\mu} e^{kz} & \frac{1}{2\mu}\left(\frac{1}{\alpha} - 1 + kz\right) e^{-kz} & \frac{1}{2\mu}\left(\frac{1}{\alpha} - 1 - kz\right) e^{kz} \\ e^{-kz} & -e^{kz} & -(1-kz)e^{-kz} & -(1+kz)e^{kz} \\ -e^{-kz} & -e^{kz} & -kze^{-kz} & -kze^{kz} \end{bmatrix} \tag{3.10}$$

with

$$\alpha = \frac{\lambda+\mu}{\lambda+2\mu}. \tag{3.11}$$

For the nth layer, (3.8) becomes

$$[A_n(z)] = [Z_n(z)]\,[K_n]. \tag{3.12}$$

The elements of the matrix $[Z_n(z)]$ are obtained from the elements of the matrix $[Z(z)]$ on replacing μ by μ_n and α by α_n. Singh and Garg (1985) have shown that

$$[A_n(z_{n-1})] = [a_n]\,[A_n(z_n)], \tag{3.13}$$

where the elements of the layer matrix $[a_n]$ are listed in Appendix I.

The continuity of the displacements u_2, u_3 and that of stresses p_{23}, p_{33} implies

$$[A_{n-1}(z_{n-1})] = [A_n(z_{n-1})]. \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$[A_{n-1}(z_{n-1})] = [a_n]\,[A_n(z_n)] \tag{3.15}$$

A repeated use of (3.15) yields

$$[A_1(0)] = [U]\,[A_p(H)], \tag{3.16}$$

where

$$[U] = [a_1]\,[a_2]\,.\,[a_{p-1}]. \tag{3.17}$$

It is clear from (3.3) that, for half-space (layer p), $B_p = D_p = 0$. Therefore,

$$[K_p] = [A_p,\ 0,\ C_p,\ 0]^T. \tag{3.18}$$

From (3.12), (3.16) and (3.18), we find

$$[V_o, W_o, S_o, N_o]^T = [E]\,[A_p, 0, C_p, 0]^T, \tag{3.19}$$

where

$$[E] = [U]\,[Z_p(H)]. \tag{3.20}$$

Equation (3.19) gives the following four equations

$$V_o = E_{11}A_p + E_{13}C_p, \tag{3.21}$$

$$W_o = E_{21}A_p + E_{23}C_p, \tag{3.22}$$

$$S_o = E_{31}A_p + E_{33}C_p, \tag{3.23}$$

$$N_o = E_{41}A_p + E_{43}C_p. \tag{3.24}$$

For given stresses at the surface $z=0$, S_o and N_o are known. From (3.23) and (3.24), we find

$$A_p = \Omega_1^{-1}(S_oE_{43} - N_oE_{33}),\quad C_p = \Omega_1^{-1}(N_oE_{31} - S_oE_{41}), \tag{3.25}$$

where

$$\Omega_1 = E_{31}E_{43} - E_{33}E_{41}. \tag{3.26}$$

For given displacements at the surface $z=0$, V_o and W_o are known. From (3.21) and (3.22), we obtain

$$A_p = \Omega_2^{-1}(V_oE_{23} - W_oE_{13}),\quad C_p = \Omega_2^{-1}(W_oE_{11} - V_oE_{21}), \tag{3.27}$$

where

$$\Omega_2 = E_{11}\, E_{33} - E_{13}\, E_{31}. \tag{3.28}$$

The field at any point of the medium can be easily calculated. As in (3.13), we have, for $z_{n-1} < z < z_n$,

$$[A_n\,(z)] = [a_n\,(z_n - z)]\;[A_n\,(z_n)], \tag{3.29}$$

where $[a_n\,(z_n - z)]$ is obtained from $[a_n]$ on replacing d_n by $z_n - z$. From (3.12), (3.15) and (3.29), we find ($p \geq 2$)

$$[A_n\,(z)] = [F]\quad [A_p,\, 0,\, C_p,\, 0]^T, \tag{3.30}$$

where

$$[F] = [a_n(z_n - z)]\;[a_{n+1}]\;[a_{n+2}]\ldots[a_{p-1}]\;[\,Z_p\,(H)]. \tag{3.31}$$

Equation (3.30) is equivalent to the following four equations

$$V\,(z) = F_{11}\,A_p + F_{13}\,C_p, \tag{3.32}$$

$$W\,(z) = F_{21}\,A_p + F_{23}\,C_p, \tag{3.33}$$

$$S\,(z) = F_{31}\,A_p + F_{33}\,C_p, \tag{3.34}$$

$$N\,(z) = F_{41}\,A_p + E_{43}\,C_p. \tag{3.35}$$

For given stresses at the surface $z = 0$, we make use of (3.4)–(3.7), (3.25) and (3.32) – (3.35) to obtain the stresses and displacements at any point of the medium. We find ($z_{n-1} < z < z_n$)

$$u_2\,(z) = \int_0^\infty [F_{11}(S_0E_{43} - N_0E_{33}) + F_{13}(N_0E_{31} - S_0E_{41})]\;\frac{1}{\Omega_1}\;\begin{pmatrix}\cos ky\\ \text{-}\sin ky\end{pmatrix}\;k\;dk, \tag{3.36}$$

$$u_3(z) = \int_0^\infty\; [F_{21}\,(S_0\,E_{43} - N_0\,E_{33}) + F_{23}\,(N_0\,E_{31} - S_0\,E_{41})]\frac{1}{\Omega_1}\begin{pmatrix}\sin\;ky\\ \cos\;ky\end{pmatrix}k\;dk, \tag{3.37}$$

$$p_{23}(z) = \int_0^\infty\; [F_{31}\,(S_0\,E_{43} - N_0\,E_{33}) + F_{33}\,(N_0\,N_{31} - S_0\,E_{41})]\;\frac{1}{\Omega_1}\begin{pmatrix}\sin\;ky\\ \text{-}\cos ky\end{pmatrix}k^2\;dk, \tag{3.38}$$

$$p_{33}(z) = \int_0^\infty\; [F_{41}\,(S_0\,E_{43} - N_0\,E_{33}) + F_{43}\,(N_0\,E_{31} - S_0\,E_{41})]\;\frac{1}{\Omega_1}\begin{pmatrix}\sin\;ky\\ \cos\;ky\end{pmatrix}\;k^2\;dk, \tag{3.39}$$

where Ω_1 is defined in (3.26).

For given displacements at surface $z = 0$, we make use of (3.4) – (3.7), (3.27) and (3.32) — (3.35) to obtain the stresses and displacements. We obtain ($z_{n-1} < z < z_n$)

$$u_2(z) = \int_0^\infty [F_{11}(V_0 E_{22} - W_0 E_{12}) + F_{12}(W_0 E_{11} - V_0 E_{21})] \frac{1}{\Omega_2} \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} k\, dk \tag{3-40}$$

$$u_3(z) = \int_0^\infty [F_{21}(V_0 E_{22} - W_0 E_{12}) + F_{22}(W_0 E_{11} - V_0 E_{21})] \frac{1}{\Omega_2} \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k\, dk \tag{3-41}$$

$$p_{23}(z) = \int_0^\infty [F_{31}(V_0 E_{22} - W_0 E_{12}) + F_{32}(W_0 E_{11} - V_0 E_{21})] \frac{1}{\Omega_2} \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} k^2\, dk \tag{3.42}$$

$$p_{33}(z) = \int_0^\infty [F_{41}(V_0 E_{22} - W_0 E_{12}) + F_{42}(W_0 E_{11} - V_0 E_{21})] \frac{1}{\Omega_2} \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k^2\, dk, \tag{3.43}$$

where Ω_2 is defined in (3.28)

3.2. Antiplane Strain

The antiplane strain problem can be solved directly. For zero body force, the displacement u_1 is harmonic. Therefore, we may assume

$$u_1 = \int_0^\infty (Ae^{-kz} + Be^{kz}) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} dk, \tag{3.44}$$

in which A and B may be functions of k. Using (2.5) and (3.44), we have

$$p_{12} = \mu \int_0^\infty (-Ae^{-kz} + Be^{kz}) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k\, dk. \tag{3.45}$$

Equations (3.44) and (3.45) may be written as

$$u_1 = \int_0^\infty U \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} dk, \quad p_{12} = \int_0^\infty T \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k\, dk, \tag{3.46}$$

where

$$\begin{bmatrix} U \\ T \end{bmatrix} = [Z(z)] \begin{bmatrix} A \\ B \end{bmatrix}, \tag{3.47}$$

$$[Z(z)] = \begin{bmatrix} e^{-kz} & e^{kz} \\ -\mu e^{-kz} & \mu e^{kz} \end{bmatrix}. \tag{3.48}$$

Following Singh and Garg (1985), we obtain

$$\begin{bmatrix} U \\ T \end{bmatrix}_{z_{n-1}} = [a_n] \begin{bmatrix} U \\ T \end{bmatrix}_{z_n}, \tag{3.49}$$

where

$$[a_n] = \begin{bmatrix} ch(kd_n) & -\mu_n^{-1} sh(kd_n) \\ -\mu_n sh(kd_n) & ch(kd_n) \end{bmatrix}. \tag{3.50}$$

Proceeding as in the case of the plane strain problem and taking $B_p = 0$, we find

$$U_0' = E_{11} A_p, \tag{3.51}$$

$$T_0 = E_{21} A_p, \tag{3.52}$$

where

$$[E] = [a_1] [a_2] \dots [a_{p-1}] [Z_p (H)]. \tag{3.53}$$

For given stress on the surface, A_p is known from (3 52). In contrast, for given displacement on the surface, A_p is known from (3.51).

The field at any point of the medium can be found as before. For given stress at $z = 0$, we obtain

$$u_1 (z) = \int_0^\infty T_0 \left(\frac{F_{11}}{E_{21}}\right) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} dk, \tag{3.54}$$

$$p_{13} (z) = \int_0^\infty T_0 \left(\frac{F_{21}}{E_{21}}\right) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k \, dk, \tag{3.55}$$

where [F] is defind as in (3.31). For given displacement at $z = 0$, we get

$$u_1 (z) = \int_0^\infty U_0 \left(\frac{F_{11}}{E_{11}}\right) \begin{pmatrix} \sin ky \\ sos ky \end{pmatrix} dk, \tag{3.56}$$

$$p_{13} (z) = \int_0^\infty U_0 \left(\frac{F_{21}}{E_{11}}\right) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k \, dk. \tag{3.57}$$

4. SPECIFIED SURFACE LOADS

4.1. Plane Strain

Normal line load

Let a normal line load P per unit length be applied at the origin to the surface $z = 0$ in the positive direction of the z—axis. The boundary conditions at $z = 0$ are

$$p_{23} = 0, \; p_{33} = -P \, \delta(y), \tag{4.1}$$

where $\delta(y)$ denotes the Dirac delta function. We use the representation

$$\delta(y) = \frac{1}{\pi} \int_0^\infty \cos ky \, dk. \tag{4.2}$$

From (3.6) and (3.7), we find that

$$S_o = 0,\ N_o = \frac{-P}{\pi k^2}, \tag{4.3}$$

and that the lower solution must be taken.

The displacements and stresses are given by (3.26), (3.36)-(3.39) and (4.3) ;

$$u_2(z) = \frac{-P}{\pi} \int_0^{\infty} \left(\frac{F_{11}\, E_{33} - F_{13}\, E_{31}}{E_{31}\, E_{43} - E_{33}\, E_{41}} \right) \frac{\sin ky}{k}\, dk, \tag{4.4}$$

$$u_3(z) = \frac{P}{\pi} \int_0^{\infty} \left(\frac{F_{21}\, E_{33} - F_{23}\, E_{31}}{E_{31}\, E_{43} - E_{33}\, E_{41}} \right) \frac{\cos ky}{k}\, dk, \tag{4.5}$$

$$p_{23}(z) = \frac{-P}{\pi} \int_0^{\infty} \left(\frac{F_{31}\, E_{33} - F_{33}\, E_{31}}{E_{31}\, E_{43} - E_{33}\, E_{41}} \right) \sin ky\, dk, \tag{4.6}$$

$$p_{33}(z) = \frac{P}{\pi} \int_0^{\infty} \left(\frac{F_{41}\, E_{33} - F_{43}\, E_{31}}{E_{31}\, E_{43} - E_{33}\, E_{41}} \right) \cos ky\, dk, \tag{4.7}$$

Shear line load

Suppose that a shear line load Q per unit length is applied at the origin to the surface $z = 0$ in the positive direction of the y — axis. The boundary conditions at $z = 0$ are

$$p_{33} = 0,\ p_{23} = -\,Q\,\delta(y). \tag{4.8}$$

Using (3.6), (3.7) and (4.2), we note that

$$S_o = \frac{-Q}{\pi k^2}, \qquad N_o = 0, \tag{4.9}$$

and that the upper solution must be taken.

The displacements and stresses at any point of the medium are found to be

$$u_2(z) = \frac{-Q}{\pi} \int_0^{\infty} \left(\frac{F_{11}\, E_{43} - F_{13}\, E_{41}}{E_{31}\, E_{43} - E_{33}\, E_{41}} \right) \frac{\cos ky}{k}\, dk \tag{4.10}$$

$$u_3(z) = \frac{-Q}{\pi} \int \left(\frac{F_{21}\, E_{43} - F_{23}\, E_{41}}{E_{31}\, E_{43} - E_{33}\, E_{41}} \right) \frac{\sin ky}{k}\, dk \tag{4.11}$$

$$p_{23}(z)= \frac{-Q}{\pi}\int_0^\infty \left(\frac{F_{31}\,E_{43}-F_{33}\,E_{41}}{E_{31}\,E_{43}-E_{33}\,E_{41}} \right) \cos ky\, dk, \tag{4.12}$$

$$p_{33}(z)= \frac{-Q}{\pi}\int_0^\infty \left(\frac{F_{41}\,E_{43}-F_{43}\,E_{41}}{E_{31}\,E_{43}-E_{33}\,E_{41}} \right) \sin ky\, dk, \tag{4.13}$$

Uniform normal load

Let a uniform normal load p per unit area act over the segment $-l \leq y \leq l$ of the y—axis in the positive direction of the z—axis. The boundary conditions at z=0 are

$$p_{23}=0, \quad p_{33}= \begin{cases} -p_0 & |y| < l, \\ 0 & |y| > l, \end{cases} \tag{4.14}$$

We may write

$$p_{33} = \frac{-2p_0}{\pi}\int_0^\infty \left(\frac{\sin\, kl}{k} \right) \cos ky\, dk. \tag{4.15}$$

From (3.6), (3.7), (4.14) and (4.15), we find that

$$S_0=0. \; N_0 = \frac{-2p_0}{\pi} \left(\frac{\sin k\, l}{k^3} \right), \tag{4.16}$$

and that the lower solution must be taken.

The displacements and stresses at any point of the medium are found to be

$$u_2(z) = \frac{-2p_0}{\pi}\int_0^\infty \left(\frac{F_{11}\,E_{33}-F_{13}\,E_{31}}{E_{31}\,E_{43}-E_{33}\,E_{41}} \right) \frac{\sin\, kl}{k^3} \sin ky\, dk, \tag{4.17}$$

$$u_3(z) = \frac{2p_0}{\pi}\int_0^\infty \left(\frac{F_{21}\,E_{33}-F_{23}\,E_{31}}{E_{31}\,E_{43}-E_{33}\,E_{41}} \right) \frac{\sin\, kl}{k^2} \cos ky\, dk, \tag{4.18}$$

$$p_{23}(z)= \frac{-2p_0}{\pi}\int_0^\infty \left(\frac{F_{31}\,E_{33}-F_{33}\,E_{31}}{E_{31}\,E_{43}-E_{33}\,E_{41}} \right) \frac{\sin\, kl}{k} \sin ky\, dk, \tag{4.19}$$

$$p_{33}(z)= \frac{2p_0}{\pi} \int_0^\infty \left(\frac{F_{41}\,E_{33}-F_{43}\,E_{31}}{E_{31}\,E_{43}-E_{33}\,E_{41}} \right) \frac{\sin\ kl}{k} \cos ky\, dk. \tag{4.20}$$

4.2. Antiplane Strain

Let the boundary condition at $z=0$ be

$$p_{13}=-R(y). \tag{4.21}$$

Using (3.46), (4.2) and (4.21), we find that

$$T_0= \frac{-R}{\pi k}, \tag{4.22}$$

and that the lower solution must be taken. From (3.54), (3.55) and (4.22), the displacement and stress at any point of the medium are found to be

$$u_1\ (z) = \frac{-R}{\pi} \int_0^\infty \left(\frac{F_{11}}{E_{21}}\right) \frac{\cos ky}{k}\ dk, \tag{4.23}$$

$$p_{13}\ (z) = \frac{-R}{\pi} \int_0^\infty \left(\frac{F_{21}}{E_{21}}\right) \cos ky\, dk. \tag{4.24}$$

5. UNIFORM HALF-SPACE

In the case of a uniform half-space ($p=1$), we shall find the stresses at any point of the medium for given surface loads.

5.1 Plan Strain

For Plane strain problem

$$[E] = [z(o)] = \begin{bmatrix} \frac{-1}{2\mu} & \frac{-1}{2\mu} & \frac{1}{2\alpha\mu} & \frac{-1}{2\alpha\mu} \\ \frac{1}{2\mu} & \frac{-1}{2\mu} & \frac{1}{2\mu}(\frac{1}{\alpha}-1) & \frac{1}{2\mu}(\frac{1}{\alpha}-1) \\ 1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 \end{bmatrix} \tag{5.1}$$

and

$$[F] = [Z(z)], \tag{5.2}$$

where $[Z(z)]$ is defined in (3.10) and we have written μ for μ_1 and α for α_1.

Normal line load

We make use of (3.10), (4.6)-(4.7), (5.1) and (5.2) to obtain the stresses. We find

$$p_{23}(z) = \frac{-Pz}{\pi}\int_0^\infty k\, e^{-kz} \sin ky\, dk$$

$$= \frac{-2P}{\pi}\left[\frac{yz^2}{(y^2+z^2)^2}\right] \tag{5.3}$$

$$p_{33}(z) = \frac{-P}{\pi}\int_0^\infty (1+kz)\, e^{-kz} \cos ky\, dk$$

$$= \frac{-2P}{\pi}\left[\frac{z^3}{(y^2+z^2)^2}\right]. \tag{5.4}$$

For the evaluation of the integrals, the standard integrals listed in Appendix II have been used. The stresses obtained here coincide with the corresponding results of Sneddon (1951; p. 409).

Shear line load

Making use of (3.10), (4.12)—(4.13), (5.1), (5.2) and Appendix II, we find

$$p_{23}(z) = \frac{-Q}{\pi}\int_0^\infty (1-kz)\, e^{-kz} \cos ky\, dk$$

$$= \frac{-2Q}{\pi}\left[\frac{y^2 z}{(y^2+z^2)^2}\right]. \tag{5.5}$$

$$p_{33}(z) = \frac{-Qz}{\pi}\int_0^\infty k\, e^{-kz} \sin ky\, dk$$

$$= \frac{-2Q}{\pi}\left[\frac{yz^2}{(y^2+z^2)^2}\right]. \tag{5.6}$$

Uniform normal load

Form (3.10), (4.19)—(4.20), (5.1), (5.2) and Appendix II, we find

$$p_{23}(z) = \frac{-2p_0 z}{\pi}\int_0^\infty e^{-kz} \sin kl\, \sin ky\, dk$$

$$= \frac{-4p_0 l}{\pi}\left[\frac{yz^2}{[z^2+(l+y)^2]\,[z^2+(l-y)^2]}\right]. \tag{5.7}$$

$$p_{zz}(z)=\frac{-2p_0}{\pi}\int_0^{\infty}\left(\frac{1+kz}{k}\right) e^{-kz} \sin kl \cos ky\, dk$$

$$=\frac{-2p_0}{\pi}\left[\tfrac{1}{2}\tan^{-1}\left(\frac{2lz}{z^2+y^2-l^2}\right)+\frac{lz\ (z^2+l^2-y^2)}{[z^2+(l+y)^2]\ [z^2+(l-y)^2]}\right].$$

$$(y^2+z^2 \geq l^2) \qquad (5.8)$$

5.2. Antiplane Strain

For antiplane strain problem,

$$[E]=[Z(0)]=\begin{bmatrix} 1 & 1 \\ -\mu & \mu \end{bmatrix}, \qquad (5.9)$$

and

$$[F]=[Z(z)] \qquad (5.10)$$

is given by (3.48). Using (3.48), (4.24), (5.9) and (5.10), we obtain

$$p_{1s}(z)=\frac{-R}{\pi}\int_0 e^{-kz} \cos ky\, dk$$

$$=\frac{-R}{\pi}\left(\frac{z}{y^2+z^2}\right). \qquad (5.11)$$

6. DISCUSSION

In Section 4, we have derived the displacements and stresses at any point of the medium caused by normal and shear loads acting on the surface of a multilayered semi-infinite medium. These results are in the form of integrals over the variable k. These integrals can be evaluated by using the method suggested by Jovanovich et al. (1974). The closed form expressions for a uniform half—space given in Section 5 can be used as a check over the numerical computation.

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APPENDIX I

The elements of the matrix $[a_n]$ are :

(11) = (33) = $\text{ch}\ (kd_n) + \alpha_n kd_n\ \text{sh}\ (kd_n)$

(12) = —(43) = $\alpha_n kd_n\ \text{ch}\ (kd_n)+(1-\alpha_n)\ \text{sh}\ (kd_n)$

(13) = $(-1/2\mu_n)\ [\alpha_n kd_n\ \text{ch}\ (kd_n)+(2-\alpha_n)\ \text{sh}\ (kd_n)]$

$(14) = -(23) = \frac{-\alpha_n}{2\mu_n} kd_n \text{ sh }(kd_n)$

$(21) = -(34) = \alpha_n kd_n \text{ ch}(kd_n) + (1-\alpha_n) \text{ sh}(kd_n)$

$(22) = (44) = \text{ch}(kd_n) - \alpha_n kd_n \text{ sh}(kd_n)$

$(24) = (1/2\mu_n) [\alpha_n kd_n \text{ ch}(kd_n) - (2-\alpha_n) \text{ sh}(kd_n)]$

$(31) = -2\alpha_n \mu_n [kd_n \text{ ch}(kd_n) + \text{sh}(kd_n)]$

$(32) = -(41) = -2\alpha_n \mu_n kd_n \text{ sh}(kd_n)$

$(42) = 2\alpha_n \mu_n [kd_n \text{ ch}(kd_n) - \text{sh}(kd_n)]$

APPENDIX-II

$(z > 0)$

(1) $$\int_0^\infty e^{-kz} \cos ky dk = \frac{z}{y^2+z^2}$$

(2) $$\int_0^\infty ke^{-kz} \sin ky\, dk = \frac{2yz}{(y^2 + z^2)^2}$$

(3) $$\int_0^\infty ke^{-kz} \cos ky\, dk = \frac{z^2 - y^2}{(y+z^2)^2}$$

(4) $$\int_0^\infty e^{-kz} \sin kl \sin ky\, dk = \frac{2l\,yz}{[z^2+(l+y)^2]\ [z^2+(l-y)^2]}$$

(5) $$\int_0^\infty e^{-kz} \sin kl \cos ky\, dk = \frac{l(z^2+l^2-y^2)}{[z^2+(l+y)^2]\ [z^2+(l-y)^2]}$$

(6) $$\int_0^\infty k^{-1} e^{-kz} \sin kl \cos ky\, dk = \tfrac{1}{2} \tan^{-1}\left(\frac{2lz}{z^2+y^2-l^2}\right)$$

In relation (6) it is assumed that $y^2 + z^2 \geq l^2$. If, However, $y^2 + z^2 < l^2$ we must add $\pi/2$ to the right-hand sid (Gradshteyn and Ryzhik, 1980, p. 492).

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