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FOOTING RESPONSE UNDER VIBRATORY LOADING

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SYNOPSIS

The dynamic response of a soil-footing system subjected to vertical vibratory loading is analyzed using the equation of motion with the kinematic and force parameters represented in phase diagram form. Large scale prototype circular footing tests on cohesive soil are analyzed. Dissipation stress function amplitude is given as a function of the strain rate amplitude and the energy storage or restoring stress amplitude is presented as a function of a nondimensional displacement amplitude. Dissipation and restoring response are both nonlinear. The physical variables considered include size and mass of the footing, static stress level, footing displacement, damping of the system, applied dynamic force, phase angle and frequency of loading. The response includes diameters ranging from 62 inches to 124 inches, weights from 12850 lbs. to 51280 lbs., applied force amplitunes between 525 lbs. and 52000 lbs., and frequencies up to the resonant values.

INTRODUCTION

The degree of complexity of many of the problems currently confronting the field of soil dynamics is such that the soil response under various loading conditions is extremely difficult to adequately estimate. Theoretical developments in the soil dynamics field, in general, have been highly restricted with regard to their applicabilty to represent actual field conditions because of general lack of basic knowledge of the response behavior of soils and soil-structure systems under a variety of loading conditions. This difficulty is due to the complexity ofsoil as a structural material and also to the complicated interaction of the soil and the structure being supported. Dynamic studies in soil mechanics seem to fall into two catagories; namely, the response of soil-structure systems and the determination of dynamic soil properties. Present knowledge of soil properties indicates that the general forms of stress-strain-time relations will probably be very complicated, nonlinear relations which may take the form of integral equations. In addition, nonlinearities may arise because of the soil-structure interaction as well as the development of finite deformations.

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The purpose of this paper is the analysis of the response of a full scale soil-footing system using the equation of motion directly, with the kinematic and force parameters represented in phase diagram form. Dissipation stress function amplitude and restoring stress function amplitude relations are given for field situations.

There are a multitude of transient and steady state dynamic problems of practical importance involving soils and soil-foundation systems which can be represented by the differential equation of motion

$$B\ddot{x} + R_1 + R_2 = F_d(t) \qquad (1)$$

where :

$\ddot{x}$ = acceleration of the dynamic system

B = a function of the mass and distribution of mass of the dynamic system

R_1 = energy dissipation function

R_2 = restoring function of the system

$F_d(t)$ = applied dynamic forcing function which is a function of time,

and the dot indicates differentiation with respect to time. Although the form of Eq. (1) is one dimensional, it can be easily written in functional form in several dimensions. The variable x is considered to be a generalized coordinate and hence symbolizes a variety of motions including a rotation, θ, about some axis.

In order to attempt to develop solutions to Eq. (1), one must know the explicit form of the function B as well as the forms of the energy dissipation function R_1 and the energy storage or restoring function R_2. The functions R_1 and R_2 are manifestations of the stress-strain-time response of the particular soil under consideration as well as the geometry involved and, hence, in general, unknown functions. It is important to note that the functions R_1 and R_2, as given in Eq. (1), may be quite general nonlinear functions and include geometry and relative mass effects. In addition, the functional B is a function of the interaction of the particular soil-structure (soil-foundation) system under consideration; that is, the soil type, type and geometry of the structure, and the type as well as magnitude of the loading. Thus, realistic theoretical solutions to Eq. (1), as well as other systems of equations applicable to various static and dynamic phenomena in soil mechanics, require a knowledge of stress-strain-time response of soils. No such relations are available at present.

Theoretical solutions of Eq. (1) have been given for highly idealized, simplified, assumed forms of R_1, R_2 and B. However, in general, these solutions are highly restricted with regard to their applicability to represent actual field conditions, and they have faild to agree in many respects with the results of experimental studies. In addition, the most extensive experimental studies are on models or relatively small footings with prototype investigations quite limited in scope. This is clearly indicated in the literature on soil dynamics.

Since the present paper is concerned with various aspects of the determination of the funcrions R_1 and R_2, the basic approach lies in the controlled utilization of the differential equation of motion itself. By testing soil-footing systems with a controlled or prescribed forcing function of time, F_d (t), and making appropriate measurements, it is possible to determine explicit values of R_1 and R_2 in Eq. (1). It would be desirable that these functionals be expressed in the form of stresses in order to obtain maximum generality and a possible correlation between various static and dynamic phenomena from both the load-deformation and stability viewpoints. By testing soil specimens in the laboratory with similarly controlled or prescribed forcing functions, it may be possible also to use the differential equation of motion to obtain the energy dissipation and restoring functions and, hence, form a possible correlation between the soil response properties, as determined by the two methods (laboratory soil test and prototype foundation test). Such a correlation might allow extrapolation of the prototype test results on specific soils to other soil types. An investigation of vibratory testing of soil specimens in the laboratory has been undertaken by the senior author with the initial phase given by Kondner (1961, 1962) and more recent results reported by Kondner, Krizek, and Haas (1963).

Another method of greatly expanding the range of practical usefulness of the prototype test programs is to expand the range of variables by conducting simple small scale model tests, designed and tested using nondimensional techniques. To insure realistic model representation, similitude of model and prototype, the actual test results from the prototype studies can be a control and check on the model study. Such a model-prototype feedback control and check system in conjunction with nondimensional techniques might greatly enhance the reliability of model methods as one of the tools of the soil-foundation field.

THEORETICAL CONSIDERATIONS

If the steady state displacement-time records of prototype footings under vibratory loading are considered to be harmonic wave forms, the displacement, x, can be written

$$x = x_0 \cos \omega t \tag{2}$$

while the applied forcing function, F_d, is given as a harmonic function of the same frequency

$$F_d = F_D \cos (\omega t + \delta) \tag{3}$$

in which

x = footing displacement at any time
x_0 = displacement amplitude
F_d = applied force at any time
F_D = force amplitude
ω = frequency of loading
δ = phase angle between the force and displacement vectors

Inherent in such a harmonic consideration is the concept of linearity, since the motion in nonlinear vibration may be periodic but not harmonic. Previous research and the soil mechanics literature indicate that soil is a nonlinear material. Nevertheless, depending upon the degree of nonlinearity, it is possible to use a harmonic approximation for the response, particularly when only amplitude data are being considered.

The displacement and force given by Eqs. (2) and (3) are simple harmonic functions of time. It is often advantageous to represent a simple harmonic function in terms of a rotating vector. In Fig. 1 the amplitude of vibratory displacement, x_o, is taken as the length of the vector and is rotated about an axis through the end of the vector and perpendicular to the plane of the paper. By uniformly rotating the vector, its projection, x, on any fixed line in the plane of the paper will change according to Eq. (2).

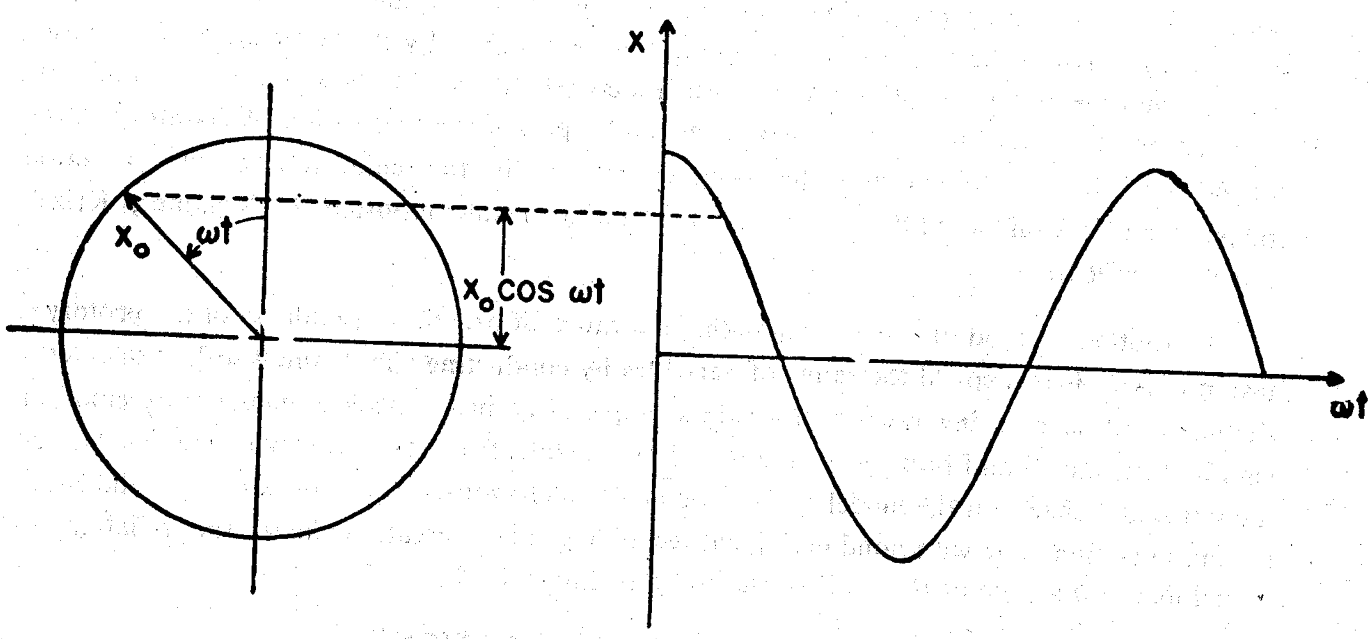

Figure 1 Rotating Vector Representation of Harmonic Motion

Utilizing constructions similar to that given in Fig. 1, it is possible to represent the various terms in Eq. (1) as rotating vectors. Consider that R_1 (t) and R_2 (t) are functions of the velocity, $\dot{x}$, and displacement, x, respectively. The resulting displacement is written

$$x = x_o \text{ cose } \omega t \quad (4)$$

while the velocity, $\dot{x}$, and acceleration, $\ddot{x}$, are written as

$$\dot{x} = \ddot{x}_o \omega \sin \omega t \quad (5)$$

$$\ddot{x} = - x_o \omega^2 \cos \omega t \quad (6)$$

Consider the special case in which the forcing function is generated by the centrifugal force due to a rototing eccentrically mounted mass. The forcing function can be written as

$$F_d = F_D \cos (\omega t + \delta) = M_o e \omega^2 \cos (\omega t + \delta) \quad (7)$$

in which M_0 is the eccentric mass , e is the eccentricity, and the force amplitude, F_D , is

$$F_D = M_0 e \omega^2 \tag{8}$$

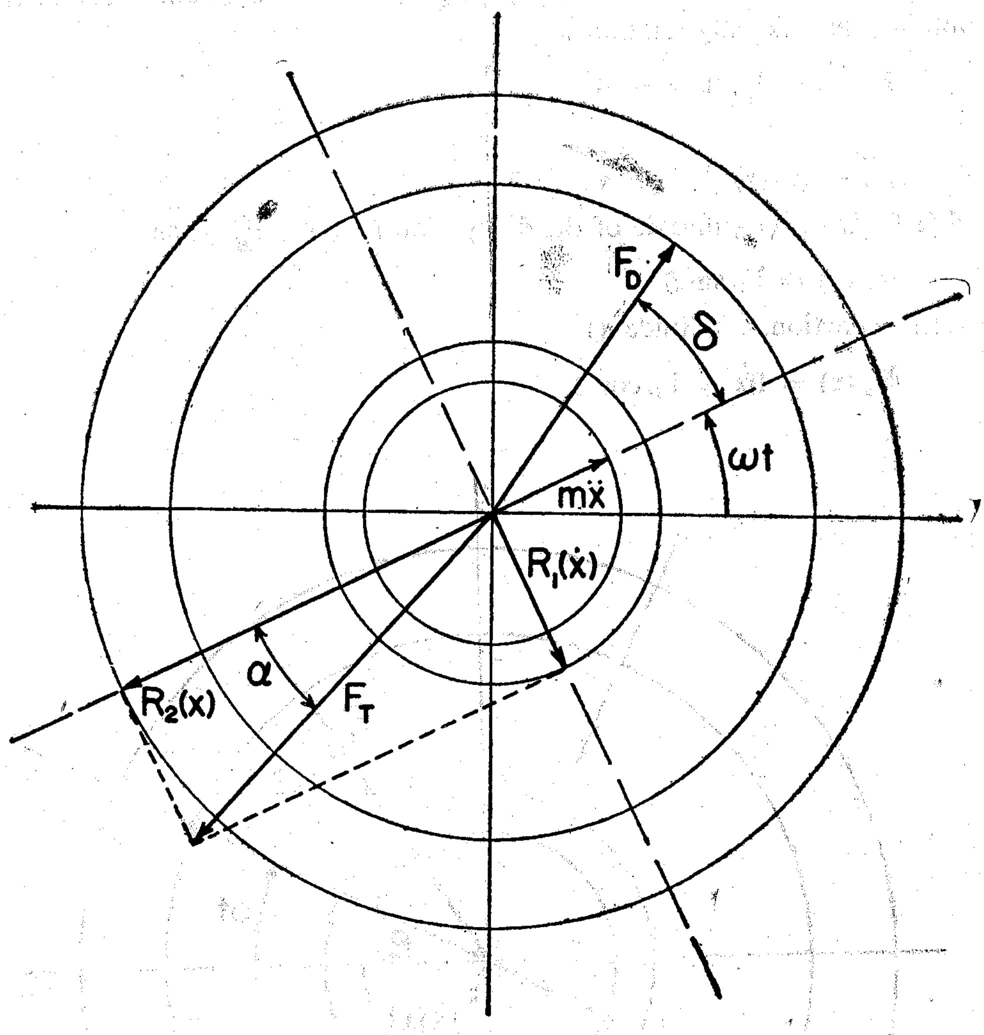

Figure 2 Phase Diagram: Kinematic Parameters and Force

Since the displacement is a cosine function, the velocity is a negative sine function and the acceleration is a negative cosine function; the velocity and acceleration are 90° and 180°, respectively, out of phase with the displacement. Fig. 2 is a vector diagram of the displacement, velocity and acceleration. The angles between the vectors are called phase angles and the diagram itself is called a phase diagram. Since all of the vectors in Fig. 2 are rotating at the same frequency, they may be considered as turning like the spokes of a wheel, preserving their relative positions in the wheel.

Using D'Alembert's principle, the inertial term $B\ddot{x}$ is a force whose direction is opposite to that of the acceleration vector. The restoring function vector $R_2(x)$ is opposite to that of the displacement and the dissipation function vector $R_1(\dot{x})$ is in the opposite direction of the velocity. Thus, the phase diagram for the force system can be constructed as given in Fig. 3

for the situation in which the forcing function F_d leads the displacement function by the phase angle δ. By resolving the forcing function into two components parallel and perpendicular to the displacement vector and then applying the equilibrium conditions at an instant of time, one obtains the following relations:

$$R_1(\dot{x}) - F_D \sin\delta = 0 \tag{9}$$

and

$$B\ddot{x} - R_2(x) + F_D \cos\delta = 0 \tag{10}$$

Eqs. (9) and (10) give the amplitude of the dissipation function $R_1(t)$ as

$$R_1(\dot{x}) = F_D \sin\delta \tag{11}$$

and the restoring function amplitude as

$$R_2(x) = B\ddot{x} + F_D \cos\delta \tag{12}$$

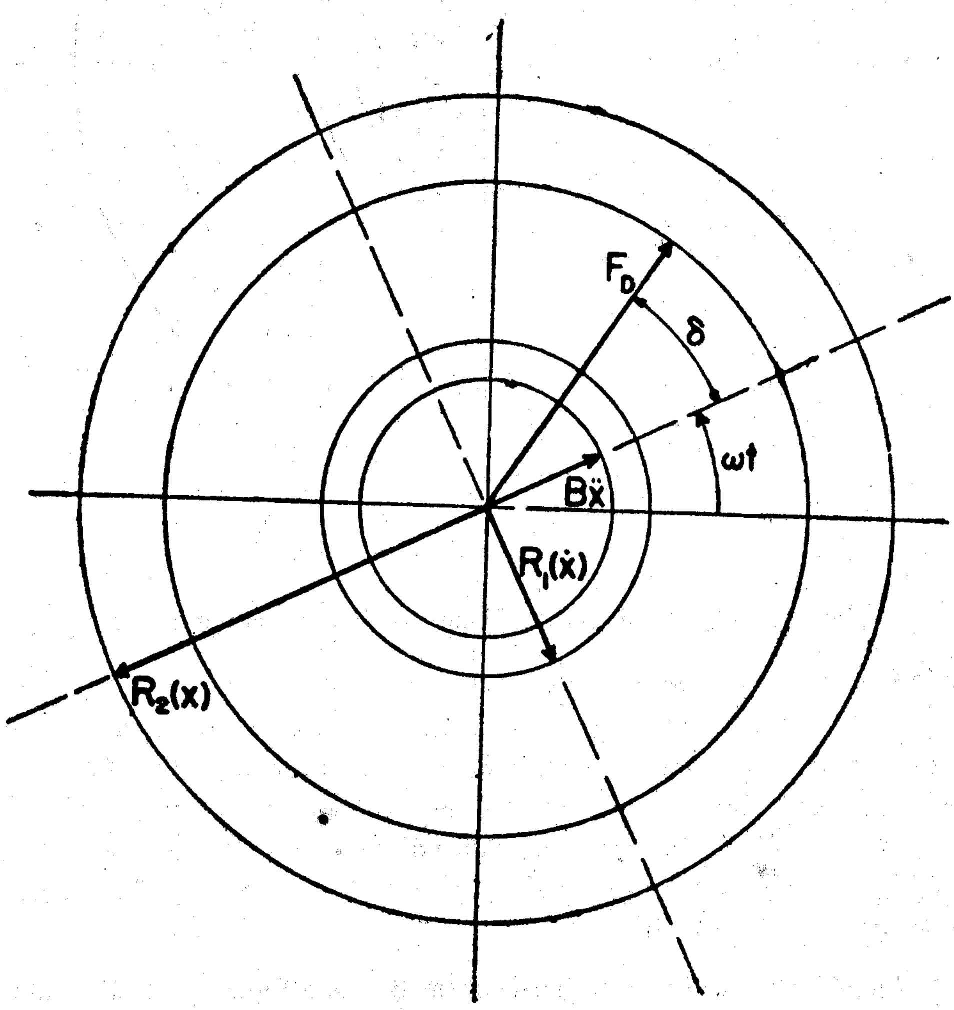

Figure 3 Phase Diagram: Force Parameters

The same relations can be obtained directly from the equation of motion. Substitution of Eqs. (6) and (3) into Eq. (1) gives

$$-B\omega^2 x_0 \cos \omega t + R_1(t) + R_2(t) = F_D \cos (\omega t + \delta) \quad (13)$$

Since $R_1(t)$ and $R_2(t)$ can be written as

$$R_1(t) = R_1(\dot{x}) \sin \omega t \quad (14)$$

and

$$R_2(t) = -R_2(x) \cos \omega t, \quad (15)$$

substitution into Eq. (13) gives

$$-B\omega^2 x_0 \cos \omega t + R_1(\dot{x}) \sin \omega t - R_2(x) \cos \omega t = F_D \cos (\omega t + \delta) \quad (16)$$

For the condition $\omega t = \pi/2$, Eq. (16) gives

$$R_1(\dot{x}) = -F_D \text{ sine } \delta, \quad (17)$$

and for the condition $\omega t = 0$, it gives

$$R_2(x) = -[F_D \sin (90^\circ + \delta) + B\omega^2 x_0] \quad (18)$$

or

$$R_2(x) = -[F_D \cos \delta + B\omega^2 x_0] \quad (19)$$

The negative signs in Eqs. (17) and (18) indicate that the vectors $R_1(\dot{x})$ and $R_2(x)$ are in the directions opposite the velocity and displacement vectors, respectively. The above relations can also be developed for lag phase angles instead of lead phase angles. In the above development, the explicit functional form of $R_1(\dot{x})$ and $R_2(x)$ have been left open; that is, specific form have not been assumed.

By Eq. (1) by a characteristic area A, one obtains

$$\frac{B\ddot{x}}{A} + \frac{R_1(t)}{A} + \frac{R_2(t)}{A} = \frac{F_d(t)}{A} \quad (20)$$

which can be written in terms of stresses as

$$\sigma_I(t) + \sigma_1(t) + \sigma_2(t) = \sigma_d(t) \quad (21)$$

where

$\sigma_I(t)$ = inertial stress as a function of time,
$\sigma_1(t)$ = dissipation stress as a function of time,
$\sigma_2(t)$ = restoring stress as a function of time,
$\sigma_d(t)$ = dynamic stressing function of time.

Dividing the dissipation and restoring functions of Eqs. (14) and (15), respectively, by the characteristic area A, one obtains

$$\sigma_1(t) = \sigma_1(\dot{x}) \sin \omega t \quad (22)$$

and

$$\sigma_2(t) = -\sigma_2(x) \cos \omega t \quad (23)$$

where $\sigma_1(\dot{x})$ and $\sigma_2(x)$ are the dissipation stress amplitude and restoring stress amplitude, respectively. These amplitudes can be written

$$\sigma_1(\dot{x}) = \frac{R_1(\dot{x})}{A} \tag{24}$$

and

$$\sigma_2(x) = \frac{R_2(x)}{A} \tag{25}$$

where $R_1(\dot{x})$ and $R_2(x)$ are given by Eqs. (11) and (12), respectively.

PROTOTYPE TEST RESULTS

The senior author has been involved in the analysis of the results of a number of prototype footing tests conducted using vertical sinusoidal forces generated by the centrifugal force due to a rotating eccentrically mounted mass. The test results analyzed in this paper were obtained from reinforced concrete circular footings 62 inches, 88 inches, 108 inches and 124 inches in diameter resting on the surface of a relatively uniform silty clay. Unfortunately, extensive soil test data are not available for the test area. Each footing was loaded symmetrically with ballast secured to the footing to a static pressure of 4·25 psi. This static pressure included the weight of the footing, weight of the vibrator and ballast load. Sinusoidal forces were applied for frequencies ranging from approximately 6 cps to 30 cps, subject to the limitations of the vibrator. This corresponded to force amplitudes, F_D, ranging from approximately 525 lbs. to 52,000 lbs., depending upon the magnitude of the eccentric mass, the eccentricity, and frequency of oscillation. All footings were carefully instrumented with various configurations of transducers and pickups for both test control and displacement measurement. Special instrumentation was used to measure the phase angle, δ, between the applied force and the footing displacement. For a footing test, a particular eccentricity was selected for a constant magnitude of eccentric mass and steady state conditions were obtained for various values of frequency. Four values of eccentricity were used for each footing. Thus, for each frequency of oscillation, the force amplitude, vertical displacement amplitude, and phase angle between force and displacement were obtained.

Fig. 4 is a graphical representation of the amplitude of the energy dissipation stress, $\sigma_1(\dot{x})$, as a function of the strain rate amplitude for the experimental program with the static stress level of 4·25 psi. The dissipation stress amplitude, $\sigma_1(\dot{x})$, was calculated by dividing the dissipation function amplitude of Eq. (11) by the footing area as indicated in Eq. (24) while the strain rate amplitude is the product of the displacement amplitude and the frequency of oscillation.

The amplitude of the restoring function is represented in Fig. 5 by the restoring stress amplitude as a function of the nondimentional displacement amplitude, x_o/d, where d is the diameter of the footing. The restoring stress amplitude, $\sigma_2(x)$, is obtained by dividing the restoring function amplitude of Eq. (12) by the footing area as indicated in Eq. (25).

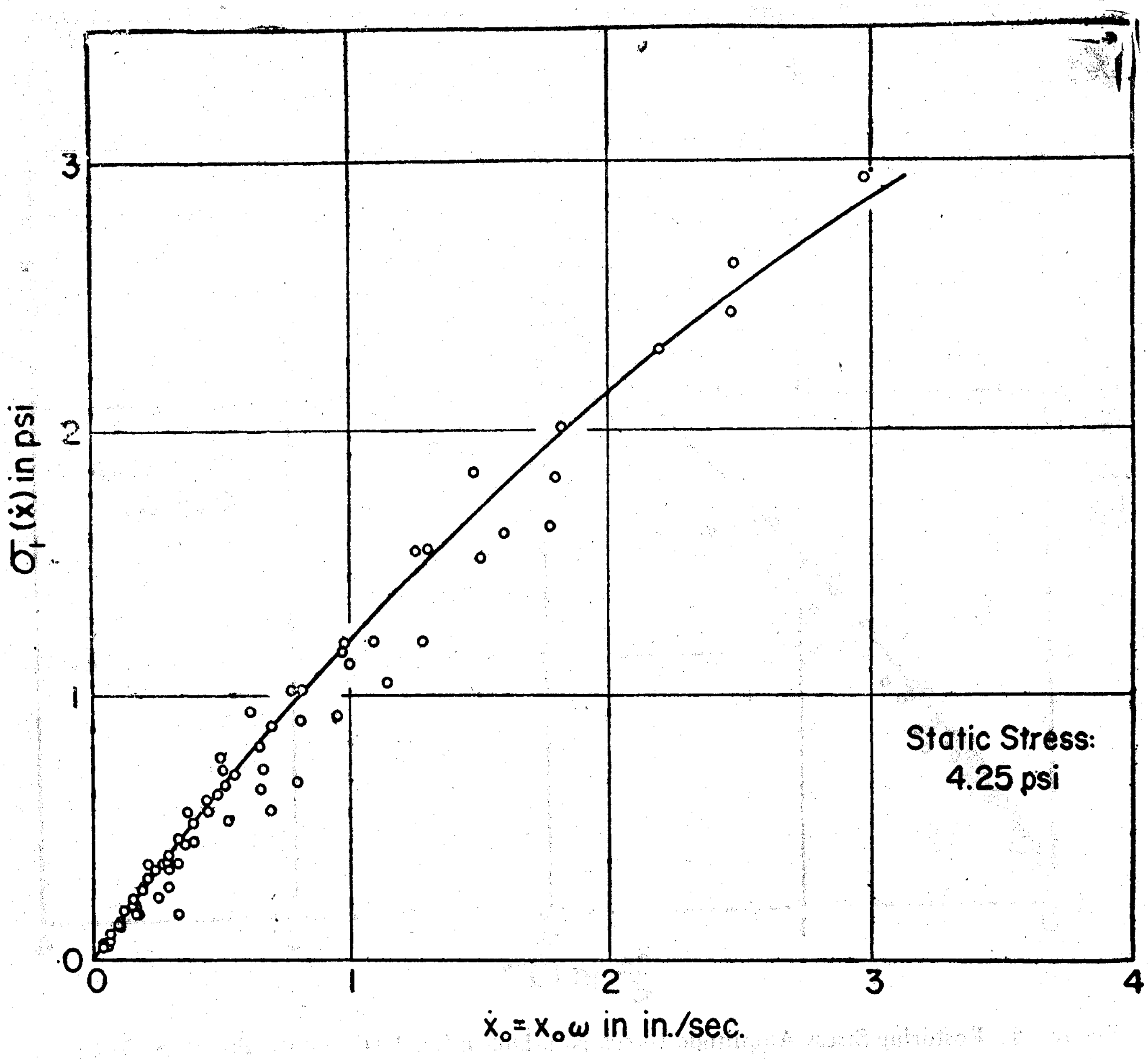

Figure 4 Dissipation Stress Amplitude versus Strain Rate Amplitude: Prototype Tests

Figs. 4 and 5 include the amplitude-frequency-phase angle data with increasing frequency up to resonant frequency. For frequencies greater than resonant frequency, both the dissipation stress amplidude and restoring stress amplitude decrease rapidly and fall below the response given in Figs. 4 and 5. It is interesting to note that Figs. 4 and 5 include the effects of varying area, static weight, frequency, eccentric setting, eccentric mass and displacement amplitude. The areas included are 20·97, 41·94, 62·92, and 83·89 sq. ft; while the static weights included 12,820, 25,640, 38,460, and 51,280 lbs., respectively.

Although a point by point linear approximation was used by virtue of the assumed

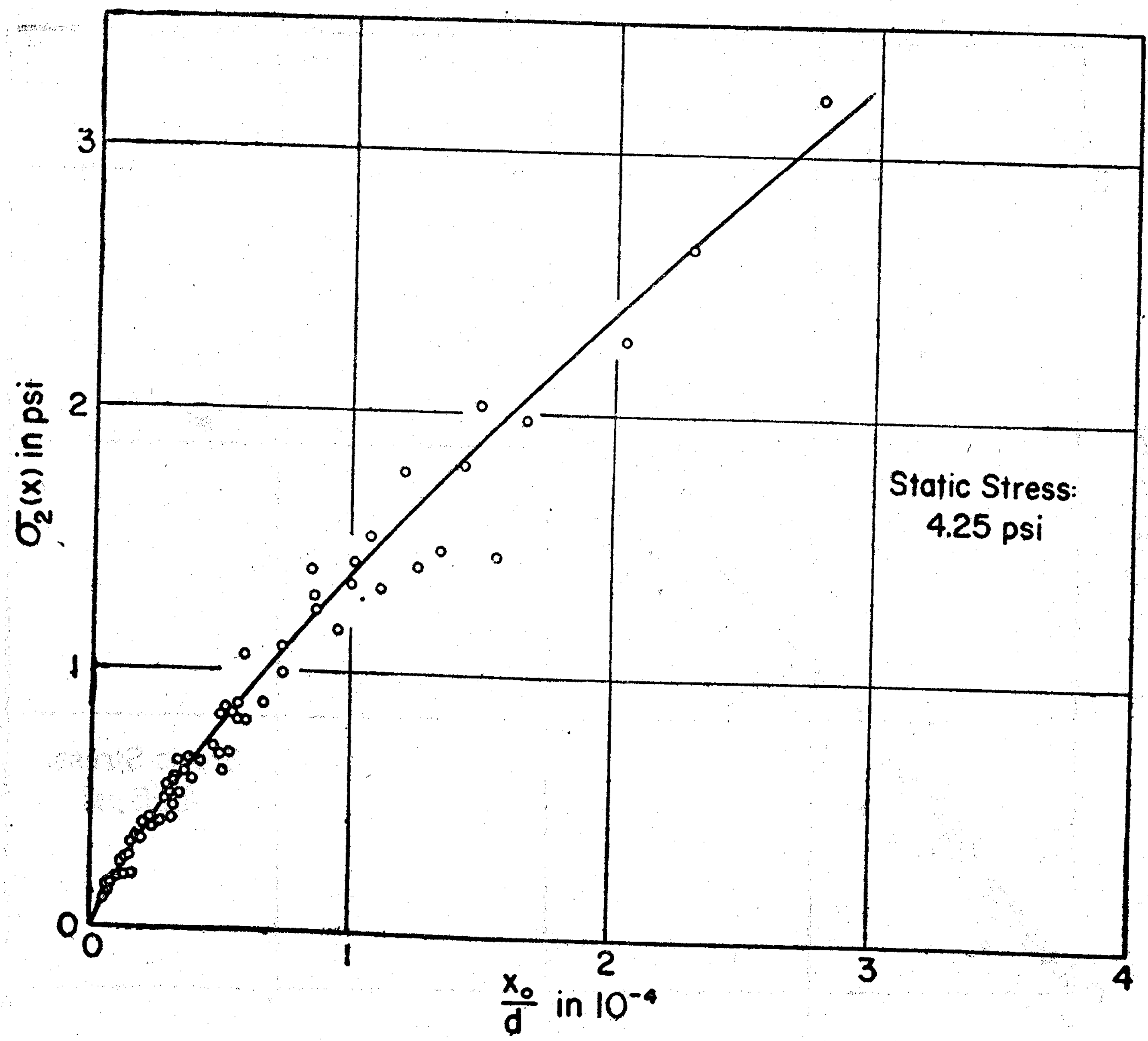

Figure 5 Restoring Stress Amplitude versus Non-Dimensional Displitude: Prototype Tests

harmonic wave form and subsequent representation with phase diagrams, the prototype footing response of Figs. 4 and 5 is definitely nonlinear. Thus, linear approximations may be useful in studying various aspects of soil-foundation response. It must be emphasized that although the present study is relatively extensive for a prototype investigation, it is quite limited in terms of the many factors that influence dynamic response of soil-foundation systems. The loading was restricted to sinusoidal and may be associated with that of interest in problems of machine vibration. Further analysis must be conducted to study the effects of size of footing, mass of the system, mode of vibration, magnitude of displacement, static stress level, magnitude and type of loading, frequency, resonant frequency, and the characteristics of the soil on which the foundation is supported. It is felt that some of these effects are responsible for the scatter in Figs. 4 and 5; hence, the necessity for a more detailed analysis.

CONCLUSIONS

Prototype response of circular footings subjected to vertical vibratory loading can be conveniently analyzed with an amplitude linear approximation of assumed harmonic motion using the equation of motion and kinematic as well as force parameters in phase diagram form. The energy storage or restoring stress amplitude can be represented as a function of a nondimentional displacement amplitude and the dissipation stress function amplitude can be related to the strain rate amplitude. Both the dissipation and restoring aspects of the cohesive soil-footing response are nonlinear. The response includes diameters ranging from 62 to 124 inches, weights from 12,820 to 51,280 lbs., applied force amplitudes between 525 to 52,000 lbs., and frequencies up to the resonant values.

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