VIBRATION ANALYSIS OF STRUCTURES BY THE ENERGY METHOD

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SYNOPSIS

This paper deals with the Energy method for determining natural frequencies of structures. The frequency or the period of vibration is calculated by comparing the potential energy of the deflected structure with its Kinetic energy as it passes through its normal static position. Rayleigh's principle, the static deflection method or a special case of Rayleigh's method, Ritz's development and Southwell Dunkerley extension of Rayleigh's principle are discussed.

A method, called the method of "Dynamic Convergence", for solution of multistoreyed framed structures, is explained. A 10-storey reinforced concrete building frame, which has been originally designed by the author for an earthquake allowance of 20% g for comparing the increase in cost due to earthquake resistant design, is analysed for its period of vibration by applying the method of Dynamic Convergence and the Energy principle. Any desired accuracy can be obtained by this method and therefore it is comparable to the other classical methods.

INTRODUCTION

In the design of an earthquake proof structure, the basic seismological data required are the natural period of vibration of the structure and the natural period of vibration of the ground. The degree to which the natural period of vibration of a structure synchronizes with the ground movements may determine whether or not the building will successfully withstand the earthquake. Again, study of vibration characteristics of a building may also be required in areas other than the earthquake zones. Thus the housing of a machine having reciprocating motion in an existing building or the design of a new building for such use might well be conditioned by a vibration study of the building. Different methods are available for calculating period of vibration of structures. In this paper Energy method for determining the period of vibration of multi-storeyed frame is discussed.

Energy Method:

The differential equation of motion for a mass m performing simple Harmonic Motion can be written as–

$$m\ddot{x} + Kx = 0 \text{ or } \ddot{x} + p^2x = 0 \ldots\ldots\ldots(1) \left[p^2 = \frac{K}{m}\right]$$

where K is the spring constant or the restoring force. This equation is satisfied for

$$x = c_1 \cos p\,t + c_2 \sin p\,t$$

whence we can see that the time period $T = \frac{2\pi}{p}$ or the frequency $f = \frac{p}{2\pi}$

Natural frequencies can sometimes be advantageously calculated by using the law of conservation of energy provided that damping is negligible. Our discussion will be based on a study of the potential and the Kinetic energies of a system in motion and their simple relation with the system's natural frequency parameter.

If we multiply equation (1) by $\dot{x}$, we have $\dot{x}\ddot{x} + \frac{k}{m} x\dot{x} = 0$. This expression lends itself to a direct integration, viz.

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}\frac{K}{m}x^2 = C$$

$$\text{or} \quad \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2 = Cm \qquad (2)$$

The first term of this expression is seen to give the instantaneous Kinetic energy of the motion of the mass, and the second term represents the instantaneous potential energy content of the linear restoring element with reference to the potential energy level required by the static equilibrium position of the system. When the displacement x is a maximum X, the velocity $\dot{x}$ is zero and all the dynamic energy of the system is potential. Similarly,

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when the displacement x is zero, the velocity $\dot{x}$ is a maximum $\dot{X}$ and all the dynamic energy of the system is Kinetic. The energy equation of a vibrating system can therefore be written as

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2 = \frac{1}{2}Kx^2 = \frac{1}{2}m\dot{x}^2 = Cm = \rho \quad (3)$$

If the characteristic deflection curve of a structure, i.e. the maximum displacement of different points along its length is known, the work done upon the structure by the oscillating masses which is the potential energy of the deflected structure can be calculated. By the law of conservation of energy, the Kinetic energy of the system as it passes through its figure of rest becomes known. Since the masses are known and the Kinetic energy has been determined, the period may be deduced.

The formal requirements of the deflection curve are that it must sattisfy the end conditions and that in between these end conditions, it must coincide with the actual dynamic deflection curve. Rayleigh (Rayleigh, 1945) has shown that the fundamental natural frequency as calculated from the assumed shape of a dynamic deflection curve of a system, will be equal to or higher than the system's true natural frequency.

Static deflection curve has got a wide application for this purpose. This is the deflection that the system would undergo under static condition if the gravity action is supposed to be acting on the masses of the system.

Better approximations in calculating the fundamental frequency and also the frequencies of higher modes of vibration can be obtained by "Ritz's method" which is a further development of Rayleigh's method (Ritz, 1911). In using this method, the equation of the deflection curve representing the mode of vibration is to be taken with several parameters, the magnitudes of which should be chosen in such a manner as to reduce to a minimum the frequency of vibration.

To apply Ritz's method in calculating the frequency f of the fundamental type of vibration, the first step is to choose a suitable expression for the deflection curve. Let $Q_1(x)$, $Q_2(x)$, be a series of function satisfying the end conditions and suitable for rep esentation of y. Then

$$y = a_1 Q_1(x) + a_2 Q_2(x) + a_3 Q_3(x) +$$

represents suitable deflection curve of the vibrating system. Taking a finite number of terms in this expression means superimposition of certain limitation on the possible shapes of the deflection curve and due to this fact the calculated frequency is usually higher than the exact value of this frequency. In order to obtain the approximation as close as possible, Ritz proposed to choose the coefficients a_1, a_2, a_3... so as to make the frequency a minimum. Thus, by equating to zero the partial derivative of the expression for frequency with respect to a_n, a system of equations homogeneous and linear in a_1, a_2, a_3 ... is obtained, the number of which is equal to the number of coefficients a_1, a_2, a_3 .... Such a system of equations can yield for a_1, a_2, a_3 ... solution different from zero only if the determinant of these equations is equal to zero. This condition brings us to the "frequency equations" from which the frequencies of the various modes of vibration can be calculated.

If a composite system can be split up into several isolated systems, we can get a lower limit to the frequency of the composite system as compared with an upper limit given by the Rayleigh approximation. This extension of Rayleigh's principle is due to Southwell and Dunkerley. In the first case, if the composite system is such that it is possible to express the total Kinetic energy in the form of one integral while the potential energy remains the sum of several integrals or terms, then the sum of the squares of the true isolated frequencies would be either less than or equal to the square of the true frequency of the composite system (Southwell 1941). In the second case, where the potential energy is contained in one term and the Kinetic energy is contributed by various inertia elements, the sum of the squares of the reciprocals of the isolated frequencies furnishes an upper limit of $1/f^2$ and consequently a lower limit of f^2, i.e. the square of the true requency of the composite system (Dunkerley, 1894). The true fundamental frequency is obtained from the higher limit given by a Rayleigh approximation and the lower limit given by a Southwell-Dunkerley approximation.

Application to Building frames :

The dynamic characteristic of a 10-storey reinforced concrete building frame is obtained by applying the energy method. The frame forms a part of an R.C.C. framed structure and has been designed by the author for Dead load, Live load, and earthquake force of 20% g using the standard code of practice given by the Indian Standard specification. The frame consists of two bays of 25′ each. The floor heights have been kept as 12′-0″. The frames have been placed 12′-0″ centre to centre.

In this problem, as before, the important part is to determine the characteristic deflection curve. To obtain the deflection curve and the dynamic forces which produce the characteristic deflection curve, a process called the "Method of Dynamic Convergence" is employed. This method is due to J.E. Goldberg and is similar to Stodola-Vianello method (Bleich, 1950).

Determination of the Deflection Curve:

Considering the bent or frame as a whole, it is clear that the shape of the deflected structure, i.e. the characteristic shape of the deflection curve of the vibrating structure, is determined by the forces exerted upon the structure by the oscillation of the various particles or masses, these forces being proportional to the products of these masses times their respective amplitudes.

In determining the shape of the deflection curve to which the structure oscillates under free vibration, use is made of physical fact that, irregular though the forces may be which initially disturb the structure from its figure of rest, the structure seeks immediately to adjust its figure to the natural deflection curve of free oscillation and with each succeeding oscillation approaches the configuration of the natural curve more closely until, finally, the natural curve is accurately matched and the previous adjusting oscillations are succeeded by self-sustaining free vibration. This fact may be proved experimentally or analytically. We may, therefore determine the shape of the natural curve by producing analytically a chain of successive circumstances closely resembling the successve aspects of the structure during its transition from the irregular ngure induced by the initial disturbance to the final deflection curve of free oscillation.

Thus we may begin by assuming an initial deflection curve for the disturbed structure which, for practical reasons, should be as correct an estimate of the final free oscillation curve as we can make conveniently. In the absence of specific data on the exact shape of the free oscillation curve, it would be convenient to assume some simple curve for the initial aspect of the disturbed and deflected structure as for example, a straight line variation of deflection with height.

For the sake of simplicity and convenience in the consideration of the specific case of building frames, it will be assumed that the masses of all the bodies and elements which make up the mass of a storey act as a single mass, concentrated at the level of the floor system.

The force exerted by a mass, m, moving with simple Harmonic motion is, at the limit of its deflection

$$\text{Force} = Cm\triangle \quad (4)$$

wharein C is a constant which applies generally to all masses of the system. A form somewhat more convenient for our purpose is obtained by transcribing equation (4)

$$\text{Relative force} = m\triangle \quad (5)$$

Assuming that each mass has exerted a relative force equal to mass times its respective initial displacements, the first adjusted deflection curve is obtained for the structure under the action of this group of assumed forces by the use of the following slope–Deflection formulae for the diflections of building frames under transverse loads. Each formula is applied, in turn, to each storey of the bent to obtain the adjusted deflection curve.

$$\theta_n = \frac{M_n + M_o}{12 \Sigma K_{gn}} \quad \ldots \quad \ldots \quad (6)$$

$$R_n = \frac{M_n}{6 \Sigma K_{en}} + \frac{\theta_m + \theta_n}{2} \quad \ldots \quad (7)$$

Where

$K_c = 2\ EI/L$ = Stiffness of column

K_g = Stiffness of girder

M = Shear force × Storey height

= Moment acting at a joint

θ	=	Angular rotation at a joint
R	=	Slope at any storey from the vertical
D	=	Relative diflection
Δ	=	Actual deflection

The above expressions for θ_n and R_n are approximate formulae. But they are simple and can be conveniently used for the present purpose. The frequency or the time period is not appreciably changed due to this approximation. Equations (6) and (7) are used for all storeys above the first. For the first storey, equations (6a) and (7a) are used assuming full restraint against rotation at the bases of the first storey columns.

$$\theta_1 = \frac{M_1+M_2}{12\Sigma K_{g1}+\Sigma K_{c1}} \quad \dots \quad (6a)$$

$$R_1 = \frac{M_1}{6\Sigma K_{c1}} + \frac{\theta_1}{2} \quad \dots \quad (7a)$$

The actual linear deflections are given by

$$D_n = (R_n)(h_n) \quad \dots \quad \dots \quad (8)$$

$$\Delta_n = \sum_{i=1}^{n} D_i \quad \dots \quad \dots \quad (9)$$

On the basis of the first adjusted deflection curve or transition curve thus determined, a new set of relative forces is computed by means of equation (5). Again using equation (6) to (9), a second adjusted deflection curve is obtained. The procedure is repeated until a deflection curve of the desired accuracy is obtained, i.e., until the free oscillation curve is determined with the desired degree of accuracy. The degree of accuracy may be tested by comparison of the successive deflection curves.

Determination of the period of vibration :

Having definitely determined the deflection curve to which the freely oscillating structure deflects and, incidentally, having determined the forces at each storey which have induced this deflection, it now becomes possible to deduce a value for the natural period of vibration. At the instant the structure has reached the end of its swing, i.e., it is fully deflected, the instantaneous velocity of each particle is zero. The total energy of the system is, therefore, entirely potential at that instant and is equal to the work done upon the structure by the decelerating particles or, what amounts to the same thing, by the relative forces previously determined.

Again, at the instant of passing through its figure of rest or normal static position, the entire energy of the structure is Kinetic and is equal to the summation of the Kinetic energies of the individual particles or masses of the system. Neglecting the internal losses due to friction and other losses of similar nature, these two quantities of energy must be equal by the law of conservation of energe.

$$\text{Total work} = \Sigma\,(\text{average force} \times \Delta) = \Sigma \left(\frac{F\Delta}{2}\right) \quad (10)$$

where, the F and Δ values used are the accepted final and correct values.

$$\text{Total Kinetic Energy} = \Sigma\left[\frac{m}{2}\left(\frac{2\pi\Delta}{T}\right)^2\right] \dots \quad (11)$$

Equating (10) and (11) and solving for T gives

$$T = 2\pi\sqrt{\frac{\Sigma\,(m\Delta^2)}{\Sigma\,(F\Delta)}} \quad \dots \quad \dots \quad (12)$$

An alterative formula, taking into account the motion of only a single mass, may be obtained as

$$T = 2\pi\sqrt{\frac{m\Delta}{F}} \quad \dots \quad \dots \quad (13)$$

Being simpler than equation (12) and entirely satisfactory for normal use, equation (13) is particularly useful as a means of checking. Equation (12), however, is more easily modified for other than the normal motions and has the further capacity of minimizing irregularities and the errors resulting thereform.

Applying the above equations, the natural period of vibration of the 10-storey reinforced concrete frame, already described, is determined analytically. A dimensioned sketch of the frame is given in Fig. (1), on which the moments of inertia (I) of the external and internal columns and of the girders are shown. The figures within bracket are corresponding values of K=2EI/L. Data and calculations are tabulated, all elements of the calculations being tabulated progressively in the order in which they are computed. The calculations for the fifth storey, whicn is typical, are carried out in detail below. The items are numbered in accordance with the numbering of the column heading of the table and will serve to explain each step of the procedure.

(1) K_c=2EI/L for each column of the designnated storey. Tabulated stiffness values are in units of 10^6lb-ft. At the fifth storey the stiffness of the two exterior

columns is 77.5 (x10⁶) lb. ft. and that of the interior column is 149 (x10⁶) lb. ft.

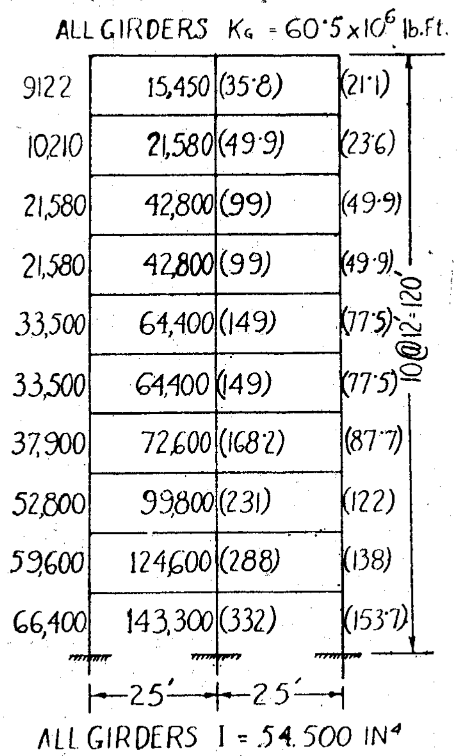

Fig. 1

(2) For the two exterior and one interior columns of the fifth storey, $6 \Sigma K_c5 = 6 \times (2 \times 77.5 + 149) = 1824$ (x 10⁶) lb. ft.

(3) The stiffness of each girder of the frame is $K_g = 60.5 \times 10^6$ lb. ft.

Therefore, for the two girders at each storey

$12 \Sigma K_g = 12 \times 2 \times 60.5 = 1452$ (x 10⁶) lb. ft.

Note that at the first storey,

$12 \Sigma K_{g1} + \Sigma K_{c1} = 1452 + 639 4 = 2091$

is tabulated for subsequent use in equation (6a), whereas the other items in this column are for use in equation (6).

(4) Mass is the mass assumed to be concentrated at the top, i.e., at the level of the girders, of the storey in question and includes the floor system, one half of the columns, walls, partitions, both above and below the floor level. The total weight of these elements divided by the acceleration of gravity is the mass for that storey.

(5) For the initial estimate of the deflection curve, a deflection of 1.00 ft. is assumed at the top of the frame with the deflection at all other points in proportion to their elevation. At the fifth storey $= 1.00 \times 60/120 = 0.500$ ft.

(6) By equation (5) the force assumed to be acting at the top of the fifth storey is relative force $= m_5 \Delta_5 = 3790 \times 0.500 = 1895$ lbs.

(7) The external shear at the fifth storey is the total of all the forces from the top of the frame down to and including the force at top of the fifth storey.

$\text{Shear}_5 = \text{Shear}_6 + \text{force } 5 = 14{,}592 + 1895 = 16{,}487$ lbs.

(8) $M_5 = \text{Shear}_5 \times \text{storey height} = 16{,}487 \times 12 = 197{,}844$ lb. ft.

(9) By eqn. (6) $\theta_5/2 = (197844 + 175{,}104/2 \times (1452) = 128.4$ Radian/10⁶

(10) $M_5/6 \Sigma K_c5 = 197{,}844/1824 = 108.5$ Radian/10_6

(11) By equation (7) $R_5 = 108.5 + 142.0 + 128.4 = 378.9$ Radian/10⁶

(12) By equation (8) $D_5 = 378.9 \times 12 = 4547$ ft./10⁶

(13) The total deflection of any storey, by equation (9), is the running total of item (12) from the first storey upto and including the storey in question. Thus $= 15854 + 4547 = 20{,}401$ ft/10⁶

(14) The deflections thus determined are scaled to a value of 1.00 ft. at the top of the frame. Thus at the fifth storey, relative deflection $= 20401/36219 = 0.563$. This item is useful chiefly in progressively checking the accuracy of the successive deflection curves.

(15) to (23) Comprise the second approximation or, in other words, determine the shape of the second transition curve. They are a repetition of the processes of (6) to (14). The second approximation corresponds very closely to the final free deflection curve and may, therefore, be used for all ordinary purposes. However, for complete convergence and a final check on the accuracy of the work, a third approximation is made.

(24) to (32) Comprise the third approximation which is made in the same manner as the first and second

Total	1	2	3	4	5	6	7	8	9	10		STOREY
	639.4	564.0	475.0	343.6	304.0	304.0	198.8	198.8	97.1	78.0	1	ΣK_c 10^6 lb. ft.
	3836	3384	2850	2062	1824	1824	1193	1193	583	468	2	$6 \Sigma K_c$ 10^6 lb. ft.
	2091	1452	1452	1452	1452	1452	1452	1452	1452	1452	3	$12 \Sigma K_g$ 10^6 lb. ft.
	3790	3790	3790	3790	3790	3790	3670	3670	3670	3510	4	Mass
	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000	5	Δ, assumed, ft.
	379	758	1137	1516	1895	2274	2569	2936	3303	3510	6	Force lbs.
	20277	19898	19140	18003	16487	14592	12318	9749	6813	3510	7	Shear lbs.
	243324	238776	229680	216036	197844	175104	147816	116988	81756	42120	8	M lb. ft.
	115.5	161	153.5	142	128.4	111.2	91.2	68.4	42.6	14.5	9	$\theta/2$ radians/10^6
	63.5	70.5	80.5	104.7	108.5	96	124	98	140.4	90.1	10	$M/6 \Sigma K_c$
	179	347	395	400.2	378.9	335.6	326.4	257.6	251.4	147.2	11	R radians/10^6
	2148	4164	4740	4802	4547	4027	3917	3091	3017	1766	12	D ft./10^6
	2148	6312	11052	15854	20401	24428	28345	31436	34453	36219	13	Δ ft./10^6
	0.059	0.174	0.305	0.438	0.563	0.675	0.782	0.869	0.950	1.000	14	Relative Δ
	224	660	1156	1660	2135	2560	2870	3190	3485	3510	15	Force lbs.
	21450	21226	20566	19410	17750	15615	13055	10185	6995	3510	16	Shear lbs
	257400	254712	246792	232920	213000	187380	156660	122220	83940	42120	17	M lb. ft.
	122.5	172.7	165	153.7	137.9	118.5	96	71	43.4	14.5	18	$\theta/2$ radians/10^6
	67	75.3	86.5	113	116.9	102.7	131	102.3	144	90.1	19	$M/6 \Sigma K_c$
	189.5	370.5	424.2	431.7	408.5	359.1	345.5	269.3	258.4	148	20	R radians/10^6
	2274	4446	5090	5180	4902	4309	4146	3232	3101	1776	21	D ft./10^6
	2274	6720	11810	16990	21892	26201	30347	33579	36680	38456	22	Δ ft./10^6
	0.059	0.175	0.307	0.441	0.570	0.682	0.789	0.873	0.954	1.000	23	Relative Δ
	224	663	1163	1672	2160	2582	2895	3200	3495	3510	24	Force lbs.
	21564	21340	20677	19514	17842	15682	13100	10205	7005	3510	25	Shear lbs.
	258768	256080	248124	234168	214104	188184	157200	122460	84060	42120	26	M lb. ft.
	123	173.5	166.1	154	138.5	119	96.3	71.1	43.5	14.5	27	$\theta/2$ radians/10^6
	67.4	75.7	87.2	113.5	117.5	103.2	131.9	102.5	144.5	90.1	28	$M/6 \Sigma K_c$
	190.4	372.2	426.8	433.6	410	360.7	347.2	269.9	259.1	148.1	29	R radians/10^6
	2285	4466	5122	5203	4920	4328	4166	3239	3109	1777	30	D ft./10^6
	2285	6751	11873	17076	21996	26324	30490	33729	36838	38615	31	Δ ft./10^6
	0.059	0.175	0.307	0.441	0.570	0.682	0.789	0.873	0.954	1.000	32	Relative Δ
24.14	0.02	0.17	0.54	1.11	1.83	2.63	3.42	4.18	4.99	5.25	33	m Δ^2
623.33	0.51	4.47	13.80	28.55	47.50	68.00	88.20	108.00	128.80	135.50	34	F Δ

$$T = 2\pi \sqrt{24.14/623.33} = 2\pi \sqrt{.0387} = 2\pi \times .1966 = 1.235 \text{ sec.}$$

$$T_5 = 2\pi \sqrt{(3790 \times .021997)/2160} = 2\pi\sqrt{.0386} = 2\pi \times .1964 = 1.235 \text{ secs.}$$

approximations. The perfect agreement between the respective quantities of (23) and (32) may be noted.

(33) The next step in the procedure is to compute the energy quantities for use in equation (12). Using the actual deflection listed in (31), again taking the fifth storey as an example,

$$m_5 \Delta_5^2 = 3790 \times (0.021996)^2 = 1.83$$

$$\Sigma (m\Delta^2) = 24.14$$

(34) Correspondingly,

$$F_5 \Delta_5 = 2160 \times .021996 = 47.50$$

$$\text{and } \Sigma (F\Delta) = 623.33$$

The procedure to use in any case is briefly :

(a) A simple deflection curve is assumed, linear, parabolic etc., depending upon the character of the structure and the distribution of the masses.

(b) The dynamic forces are calculated by means of equation (5).

(c) The deflections induced by the assumed dynamic forces are determined using the appropriate deflection calculating method.

(d) The structure is passed through the necessary number of cycles of dynamic convergence, to determine the true free vibration deflection curve, by repeating steps (b) and (c)

(e) The period is calculated by means of equation (12)

CONCLUSION

(i) From the above discussion it may be seen that the method of Dynamic convergence with Energy principle can be used, with great advantage in the analysis of multistoreyed building frames for dynamic loading.

(ii) Instead of the Dynamic convergence method, the static deflection curve can also be used. But then the dynamic character of the problem is not acknowledged. Because, actually the dynamic forces which produce the deflection curve are proportional not simply to the various masses but to the product of mass and acceleration of each element of the structure.

(iii) Compared to other methods of frequency determination, Energy Method is more general and can be made to suit any given condition by incorporating different losses that may come in actual structures.

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