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COUPLED VIBRATIONS OF A CANTILEVER BEAM OF LINEARLY VARYING CHANNEL CROSS SECTION UNDER HARMONIC EXCITATION

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Introduction

The analysis presented in this paper considers coupled vibrations of a cantilever beam of linearly varying channel cross section excited by the periodic motion of its supporting base. The harmonic supporting base motion acts in the direction normal to the beam span as shown in Fig. 1 and the shear centre of each cross — section of the beam does not coincide with the centre of gravity, consequently the torsional and bending oscillations are 'coupled'. These coupled vibration problems have received less attention so far as compared to uncoupled one and the study of such simultaneous excitation of two modes, each oscillating steadily at its own natural frequency, may be of considerable interest in vibration testing of actual structures. The method based on Rayleigh's quotient has been used to obtain the frequencies of vibrations and the accuracy of the coupled frequency is investigated by considering the application of the method to the uncoupled vibration. As the cross—section of the beam varies S_x, I_x, J_x, I_{ox} are functions of x.

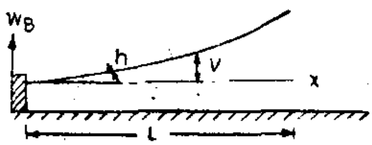

Fig. 1 Geometry of system

The Differential Equations

The differential equation for the deflected form of the neutral axis of a bar according to the elementary theory of bending is

$$E \frac{\partial^2}{\partial x^2}\left(I_x \frac{\partial^2 v}{\partial x^2} \right) = u_x \tag{1}$$

Where I_x and u_x are the moment of inertia and the intensity of the distributed load at x. E and v are the modulus of rigidity and the deflection of the beam.

If the load is distributed along the centroidal axis, the given load can be replaced by the same load distributed along the shear-centre axis, and a torque of intensity $u_x\, \delta_x$ distributed along the same axis, where δ_x is the distance between shear centre axis and centroidal axis at the point x.

Let the x-axis coincide with the shear-centre axis. Since the torsion is not uniform, the relation between the variable T and the angle of twist θ is given by Timoshenko (1955).

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$$T = GJ_x \frac{\partial\theta}{\partial x} - C_x \frac{\partial^3\theta}{\partial x^3}$$

Where GJ_x is the torsional rigidity at the point x and C_x is the warping rigidity at the point x. Differentiation of this equation with respect to x gives.

$$G \frac{\partial}{\partial x}\left\{J_x \frac{\partial\theta}{\partial x}\right\} - \frac{\partial}{\partial x}\left\{C_x \frac{\partial^3\theta}{\partial x^3}\right\} = u_x\delta_x \tag{2}$$

with the positive torque taken as shown in Fig. 2.

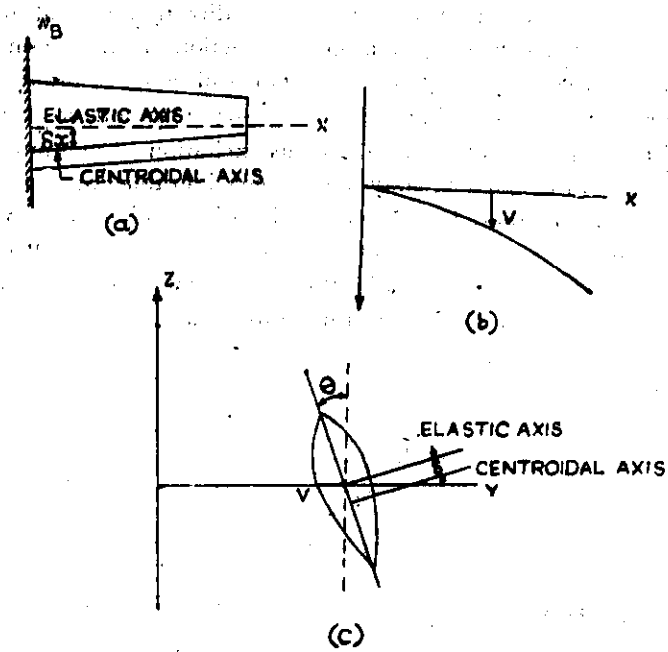

Fig. 2 Deflection and rotation of a cantilever beam from and about elastic axis

For a vibrating bar the intensity of the inertia force is $-\rho S_x \frac{\partial^2}{\partial t^2}\left(v + \delta_x\theta\right)$

where ρ is the density and S_x the area of cross-section at x and the intensity of the inertia moment about the x-axis is

$$- I_{\theta x} \frac{\partial^2\theta}{\partial t^2}$$

$I_{\theta x}$ is the mass moment of inertia at x.

The following differential equations for the coupled bending and torsional vibrations are obtained by replacing the statical load in equations (1) and (2) by the inertia forces.

$$E \frac{\partial^2}{\partial x^2}\left\{I_x \frac{\partial^2 v}{\partial x^2}\right\} = -\rho s_x \frac{\partial^2}{\partial t^2}\left(v + \delta_x\theta\right) \tag{3 a}$$

$$G \frac{\partial}{\partial x} \left\{ J_x \frac{\partial \theta}{\partial x} \right\} - \frac{\partial}{\partial x} \left\{ C_x \frac{\partial^3 \theta}{\partial x^3} \right\} = \rho S_x \delta_x \frac{\partial^2}{\partial t^2} (v + \delta_x \theta) + I_{\theta x} \frac{\partial^2 \theta}{\partial t^2} \qquad (3\ b)$$

The right hand side of equation (3a) represents an inertia loading and by considering the base motion W_B Tseng and Dugundji (1970) the governing differential equations then become

$$\left.\begin{aligned} E \frac{\partial^2}{\partial x^2} \left\{ I_x \frac{\partial^2 v}{\partial x^2} \right\} &= - \rho S_x \frac{\partial^2}{\partial t^2} (v + \delta_x \theta) - \rho S_x \frac{\partial^2 W_B}{\partial t^2} \\ G \frac{\partial}{\partial x} \left\{ J_x \frac{\partial \theta}{\partial x} \right\} - \frac{\partial}{\partial x} \left\{ C_x \frac{\partial^3 \theta}{\partial x^3} \right\} &= \rho S_x \delta_x \frac{\partial^2}{\partial t^2} (v + \delta_x \theta) + I_{\theta x} \frac{\partial^2 \theta}{\partial t^2} \end{aligned}\right\} \qquad (4)$$

where,

$$I_x = I_o \left(1 - \wedge \frac{X}{L}\right)^3, \qquad J_x = J_o \left(1 - \wedge \frac{X}{L}\right)$$

$$C_x = C_o \left(1 - \wedge \frac{X}{L}\right)^5, \qquad I_{\theta x} = I_{\theta o} \left(1 - \wedge \frac{X}{L}\right)^3, \quad \wedge = \left[1 - \frac{S_L}{S_o}\right]$$

Determination of Natural Frequencies

For all systems subject to harmonic excitation, the transient vibrations die out within a matter of short time, leaving only the steady state vibrations. Thus in studing forced oscillations the forcing function should be periodic and when steady state conditions are reached, the frequency of the motion of the system become the same as that of the forcing function. So the solutions of equation (4) are taken of the form

$$\left.\begin{aligned} v(x,t) &= A f_1(x) \; e^{i\omega t} \\ \theta(x,t) &= B \phi_1(x) \; e^{i\omega t} \\ W_B &= A F_0 \; e^{i\omega t} \end{aligned}\right] \qquad (5)$$

where AFo is the forcing amplitude, ω the frequency of vibration and A and B are constants which are not independent. $f_1(x)$ and $\phi_1(x)$ are functions of x only.

The functions $f_1(x)$ and $\phi_1(x)$ satisfy all the boundary conditions of the beam which are as follows :

$$\left.\begin{aligned} v(0) &= \frac{\partial v}{\partial x}(0) = \theta(0) = \frac{\partial^2 \theta}{\partial x^2}(0) = 0 \\ \frac{\partial^2 v}{\partial x^2}(1) &= \frac{\partial^3 v}{\partial x^3}(1) = \frac{\partial \theta}{\partial x}(1) = \frac{\partial^3 \theta}{\partial x^3}(1) = 0 \end{aligned}\right] \qquad (6)$$

Substitution of (5) in (4) gives

$$\left[E \frac{d^2}{dx^2}\left\{I_x \frac{d^2 f_1}{dx^2}\right\} - \rho S_x \omega^2 f_1 - \rho S_x F_0 \omega^2\right] A - \rho S_x \delta_x \omega^2 \phi_1 \; B = 0$$

$$\left[G \frac{d}{dx}\left\{J_x \frac{d\phi_1}{dx}\right\} - \frac{d}{dx}\left\{Cx \frac{d^3 \phi_1}{dx^3}\right\} + \rho S_x \delta^2_x \omega^2 \phi_1 + I\theta_x \omega^2 \phi_1\right] B + \rho S_x \delta_x \omega^2 f_1 \; A = 0 \qquad (7)$$

The eqns. are now put in terms of dimensionless variables $\xi = \frac{x}{L}$, $f = \frac{f_1}{L}$, $\phi = \frac{\phi_1}{L}$ where L is the length of the beam and the substitutions

$$\beta^2 = \frac{E I_0}{\rho S_0 L^4},\ \gamma^2 = \frac{G J_0}{\rho S_0 L^2},\ C_1 = \frac{C_0}{\rho S_0 L^4},\ I_\theta' = \frac{I_{\theta 0}}{\rho S_0}$$

are used.

Equations (7) become

$$\Big[\beta^2 \Big\{(1-\wedge\xi)^3 \frac{d^4 f}{d\xi^4} - 6\wedge(1-\wedge\xi)^2 \frac{d^3 f}{d\xi^3} + 6\wedge^2 (1-\wedge\xi) \frac{d^2 f}{d\xi^2}\Big\}$$

$$- (1-\wedge\xi)\Big\{\omega^2 f + \frac{F_0}{L}\omega^2\Big\}\Big] A - \delta_0 (1-\wedge\xi)^2 \omega^2 \phi B = 0 \qquad (8)$$

$$\delta_0 (1-\wedge\xi)^2 \omega^2 f A + \Big[\gamma^2 \Big\{(1-\wedge\xi)\frac{d^2\phi}{d\xi^2} - \wedge\frac{d\phi}{d\xi}\Big\} - C_1$$

$$\Big\{(1-\wedge\xi)^5 \frac{d^4\phi}{d\xi^4} - 5\wedge(1-\wedge\xi)^4 \frac{d^3\phi}{d\xi^3}\Big\} + (I_\theta' + \delta_0^2)(1-\wedge\xi)^3 \phi\,\omega^2\Big] B = 0$$

For an approximate determination of the fundamental frequency $f(\xi)$ is chosen as the shape function for the fundamental mode of uncoupled bending vibration and $\phi(\xi)$ as the shape function for the fundamental mode of uncoupled torsional vibration of a uniform cantilever beam. These shape functions satisfy the boundary conditions (6) and are

$$f(\xi) = \cosh \lambda\xi - \cos\lambda\xi - \sigma_r (\sinh\lambda\xi - \sin\lambda\xi)$$

and $$\phi(\xi) = \sin \frac{\pi}{2}\xi$$

Where, $$\lambda = 1.87510$$

$$\sigma = 0.73410$$

Equation (8) can be solved for ω^2 but the result is a function of ξ, and f and ϕ are not the exact shape function. This difficulty can be overcome Fung (1955) by multiplying the first and the second of eqns. (8) by f and ϕ, respectively, and integrating with respect to ξ from 0 to 1. The method results in a familiar Rayleigh quotient Collatz (1960) when applied to uncoupled problems, and is an extension of Rayleigh's method to coupled problem. The following equations are obtained :

$$[a_1 + a_2 + a_3 - (a_4 + a_5)\omega^2] A - a_6 \omega^2 B = 0 \qquad (10)$$

$$- a_7 \omega^2 A + [a_8 + a_9 + a_{10} + a_{11} - a_{12}\omega^2] B = 0$$

Where,

$$a_1 = \beta^2 \int_0^1 (1-\wedge\xi)^3 \frac{d^4 f}{d\xi^4} f d\xi,\quad a_2 = -6\wedge\beta^2 \int_0^1 (1-\wedge\xi)^2 \frac{d^3 f}{d\xi^3} f d\xi$$

$$a_3 = 6\wedge^2\beta^2 \int_0^1 (1-\wedge\xi)\frac{d^2 f}{d\xi^2} f d\xi,\quad a_4 = \int_0^1 (1-\wedge\xi) f^2 d\xi$$

$$a_5 = \frac{F_0}{L} \int_0^1 (1-\Lambda\xi)\, f d\,\xi, \qquad a_6 = a_7 = \delta_0 \int_0^1 (1-\Lambda\xi)^2 f \phi\, d\xi$$

$$a_8 = -\gamma^2 \int_0^1 (1-\Lambda\xi) \frac{d^2\phi}{d\xi^2}\, \phi\, d\xi$$

$$a_9 = \Lambda\, \gamma^2 \int_0^1 \frac{d\phi}{d\xi}\, \phi\, d\,\xi$$

$$a_{10} = C_1 \int_0^1 (1-\xi\Lambda)^5 \frac{d^4\phi}{d\xi^4}\, \phi\, d\xi$$

$$a_{11} = -5\,\Lambda\, C_1 \int_0^1 (1-\Lambda\xi)^4 \frac{d^3\phi}{d\xi^3}\, \phi\, d\xi$$

$$a_{12} = (I_0' + \delta_0^2) \int_0^1 (1-\Lambda\xi)^2 \phi^2\, d\xi$$

For a non trivial solution A and B must not both vanish, consequently the determinant of the co-efficients of eqns. (10) must be zero :

$$\begin{vmatrix} a_1 + a_2 + a_3 - (a_4 + a_5)\,\omega^2 & -a_6\,\omega^2 \\ -a_7\,\omega^2 & a_8 + a_9 + a_{10} + a_{11} - a_{12}\,\omega^2 \end{vmatrix} = 0$$

$$\text{i.e.}\quad p\,\omega^4 - Q\,\omega^2 + R = 0 \tag{11}$$

Where:

$$P = (a_4 + a_5)\, a_{12} - a_6\, a_7$$

$$Q = (a_1 + a_2 + a_3)\, a_{12} + (a_8 + a_9 + a_{10} + a_{11})\,(a_4 + a_5)$$

$$R = (a_1 + a_2 + a_3)\ (a_8 + a_9 + a_{10} + a_{11})$$

The solution of equation (11) is

$$\omega^2 = \frac{Q \pm \sqrt{Q^2 - 4\,PR}}{2\,P} \tag{12}$$

The right-hand side of equation (12) is positive since it may be shown that $Q^2 - 4\,PR > 0$.

The smaller of the two values of ω^2 given by eqn. (12) is an upper bound for the frequency of the fundamental mode of vibration. The larger of the two ω^2 values is an upper bound for the next higher mode of vibration.

Numerical Example

A numerical example for the coupled torsional vibrations of a beam of linearly varying cross-seetion is now presented. The frequencies are computed from eqn. (12) and the cross-section of the blade is taken as a channel section of height b_0 breadth $2b_0$ and thickness h as shown in Fig. 3. The physical constants are assumed as follows :

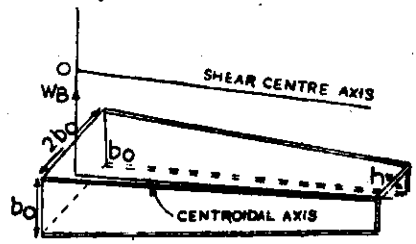

Fig. 3 Beam of linearly varying channel cross section

$$L = 10 \text{ in}, \quad E = 30\times10^6 \text{ lb/in}^2, \quad G = 12\times10^6 \text{lb/in}^2, \quad b_0 = .5 \text{ in},$$

$$h = .25 \text{ in.} \quad \rho = .27793 \text{ lb/cm in}, \quad S_x = 4b_0h\,(1-\Lambda\xi)$$

$$J_x = \frac{4}{3}\, b_0h^3\,(1-\Lambda\xi), \quad C_x = \frac{7}{24}\, hb_0^5\, E\,(1-\Lambda\xi)^5$$

$$I_x = \frac{115}{12}\, hb_0^3\,(1-\Lambda\xi)^3, \quad I\theta_x = \frac{115}{12}\, \rho hb_0^3\,(1-\Lambda\xi)^3$$

$$\delta_x = \frac{5}{8}\, b_0\,(1-\Lambda\xi)$$

With these values the fundamental frequencies for various ratios of S_L/S_0 and F_0/L have been tabulated in Table No. 1 and 2. In Fig. 4 $\omega/\omega\,\frac{Fo}{L} = 0$ has been plotted as a function of S_L/So where $\omega F_0/L = 0$ is calculated for $S_L/So = 1$, and $\omega Fo/L = 0$ is the frequency in the absence of forcing amptiude. In Fig. 5 $\omega/\omega\,\frac{F_0}{L} = 0$ has been plotted as a function of F_0/L.

Table 1

$\frac{S_L}{S_0}$	Square of the fundamental frequency for fixed $F_0/L = 2.0$	
	First root of frequency ω_1^2 Sec^{-2}	Second root of frequency ω_2^2 Sec^{-2}
0	27.37491×10^4	58.26221×10^4
0.25	16.43683×10^4	21.90009×10^4
0.50	8.02664×10^4	13.39254×10^4
0.75	4.36277×10^4	7.19757×10^4
1.00	2.59004×10^4	4.08832×10^4
1.25	1.87260×10^4	2.89634×10^4
1.50	1.53332×10^4	2.70978×10^4

Table 2

$\frac{F_0}{L}$	Square of the fundamental frequency for fixed $S_l/S_0 = 0.5$	
	First root of frequency ω_1^2 Sec^{-2}	Second root of frequency ω_2^2 Sec^{-2}
0	8.72041×10^4	14.08213×10^4
1	8.48288×10^4	13.85612×10^4
2	8.02664×10^4	13.39254×10^4
3	7.23920×10^4	11.56159×10^4
4	6.29253×10^4	9.52611×10^4
5	5.44725×10^4	7.22842×10^4
6	4.76475×10^4	4.92870×10^4

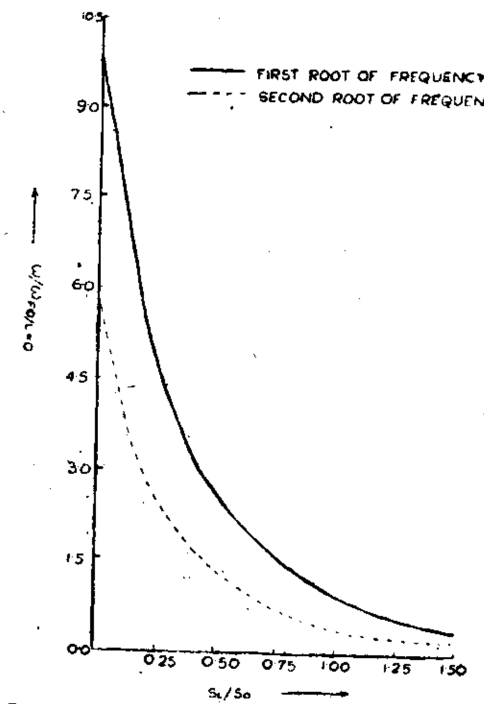

FIG.4 EFFECT OF CROSS SECTION ON FREQUENCY

Discussion and Concluding Remarks

For the uncoupled case (i.e. $\delta_x = 0$), eqn. (8) are two separate eigenvalue problems that satisfy the conditions of self-adjointness and full definitness collatz (1960). The first of eqn. (8) when $\delta_x = 0$ is

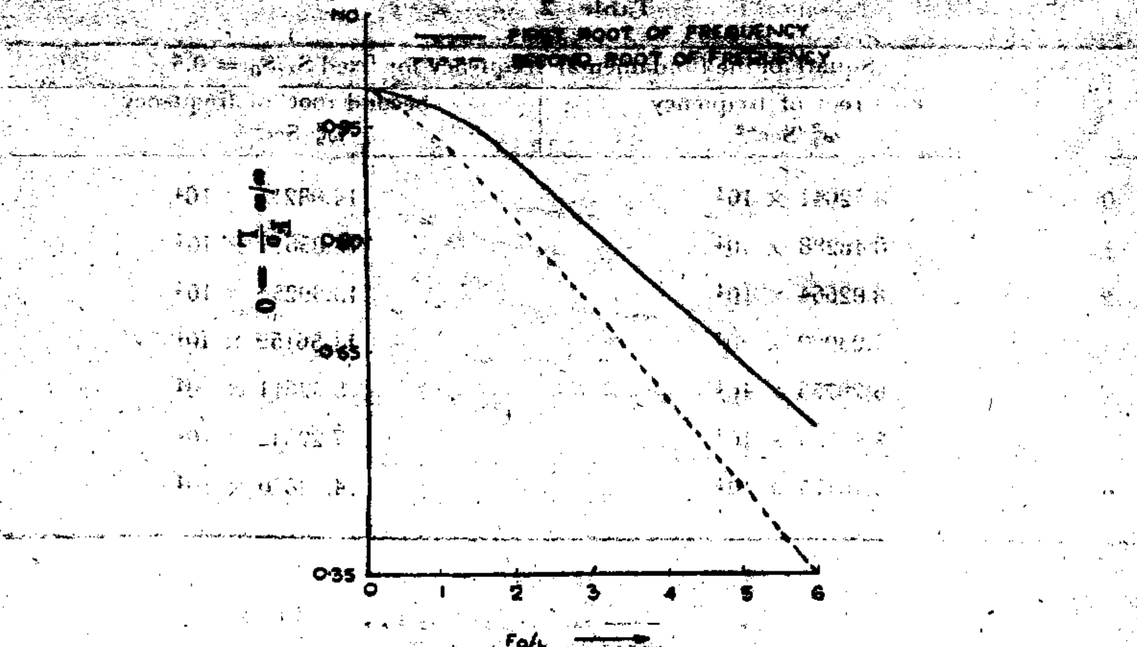

Fig. 5 Effect of forcing amplitude on frequency

$$\beta^2\left[(1-\Lambda\xi)^3\,\frac{d^4f}{d\xi^4}-6\,\Lambda\,(1-\Lambda\xi)^2\,\frac{d^3f}{d\xi^3}+6\,\Lambda^2\,(1-\xi\Lambda)\,\frac{d^2f}{d\xi^2}\right]$$
$$=(1-\Lambda\xi)\left\{\omega^2 f+\frac{F_0}{L}\,\omega^2\right\}$$

With boundary conditions $f(0)=f'(0)=f''(1)=f'''(1)=0$

This can be written as

$$U(f)=\omega^2\,V(f)$$

Where U and V are differential operators and the boundary conditions as

$$B(f)=0$$

A sequence of functions f_1, f_2, f_3 ...... can be obtained from arbitrary f_0 and application of the boundary value problems

$$U(f_k)=V(f_{k-1})$$
$$B(f_k)=0 \qquad k=1, 2, 3, \ldots\ldots$$

If the ratio f_{k-1}/f_k tends to constant as k increases, the sequence converges, and the function f_k converges to the form of the first eigen-function f. The quotient $U(f_k)/V(f_k)$ should provide an approximation to ω_e^2, where ω_e is first eigenvalue, but as it is still a function of ξ the numerator and denominator are each multiplied by f_k and integrated with respect to ξ from 0 to 1, to give Rayleigh's quotient which is an upper bound for ω_e^2, that is

$$\wedge_1 = \frac{\int_0^1 f_k\, U(f_k)\, d\xi}{\int_0^1 f_k\, V(f_k)\, d\xi} = \frac{\int_0^1 f_k\, V(f_{k-1})\, d\xi}{\int_0^1 f_k\, V(f_k)\, d\xi}$$

The form of the integrals appearing in the above expressions for $\wedge_1$, may be denoted by

$$b_{2k-1} = \int_0^1 f_k\, V(f_{k-1})\, d\xi$$

$$b_{2k} = \int_0^1 f_k\, V(f_k)\, d\xi \quad (k = 0, 1, 2 \ldots)$$

$$b_{2k+1} = \int_0^1 f_{k+1}\, V(f_k)\, d\xi$$

and are called Schwarz constants Collatz (1960) and $\wedge_1$, is the quotient of two successive Schwarz constants

$$\wedge_1 = \mu_{2k} = \frac{b_{2k-1}}{b_{2k}}$$

The quotients $\mu_{2k-1} = b_{2k-2}/b_{2k-1}$, $\mu_{2k} = b_{2k-1}/b_{2k}$, $\mu_{2k+1} = b_{2k}/b_{2k+1}$ ... ... are known as Schwarz's quotients collatz (1960) and the even numbered quotients are identical with Rayleigh's quotients.

Bounds for the first eigenvalue ω_s^2 can be obtained from the quotients μ_k, μ_{k+1} and l_2 where l_2 is the lower bound for the second eigenvalue and such that $l_2 > \mu_{k+1}$. The bounds are given by Collatz (1960).

$$\mu_{k+1} - \frac{\mu_k - \mu_{k+1}}{\frac{l_2}{\mu_{k+1}} - 1} \leqslant \omega_s^2 \leqslant \mu_{k+1} \quad (k = 1, 2, 3, \ldots \ldots)$$

For the numerical computation of the quotients f_1 is taken as the shape function for the free vibration of a beam of linearly varying cross-section, that is

$$f_1 = \cosh 1.8751\,\xi - \cos 1.8751\,\xi - 0.7341\,(\sinh 1.8751\,\xi - \sin 1.8751\,\xi)$$

and is a non-zero function and satisfies all boundary conditions and possesses continuous derivatives.

Also, a function f_0 such that

$$U(f_1) = V(f_0)$$

or $$\beta^2 \left[(1-\Lambda\xi)^2 \frac{d^4f}{d\xi^4} - 6\,\Lambda\,(1-\Lambda\xi)\frac{d^3f}{d\xi^3} + 6\,\Lambda^2 \frac{d^2f}{d\xi^2} \right] = f_0$$

can be readily obtained.

In this special eigenvalue problem with V (f) the condition

$$\int_0^1 \left[f_0\,V\,(f_1) - f_1\,V\,(f_0) \right] d\xi = 0$$

is satisfied and according to Collatz (1960) f_0 need not satisfy any boundary conditions.

Using the same numerical values for the physical constants of the beam already considered but with $\delta_z = 0$ and for $S_l/S_0 = .75$ Schwarz constants and quotients are

$$b_0 = \int_0^1 f_0^2\, d\xi \quad = 51.12523 \times 10^8$$

$$b_1 = \int_0^1 f_0\, f_1\, d\xi \quad = 11.43309 \times 10^4$$

$$b_2 = \int_0^1 (1-\Lambda\xi) \left(f_1 + \frac{F_0}{L} \right) f_1\, d\xi = 2.56314$$

$$\mu_1 = \frac{b_0}{b_1} = 4.47169 \times 10^4$$

$$\mu_2 = \frac{b_1}{b_2} = 4.46058 \times 10^4$$

and an upper bound is

$$\mu_1 \geqslant \mu_2 > \omega^2 \text{ i.e. } 4.47169 \times 10^4 > 4{:}46058 \times 10^4 > \omega^2$$

For calculating lower bound from the expression $\mu_2 - (\mu_1 - \mu_2)/(l_2/\mu_2 - 1), < \omega_0^2$, it is not essential to have a close lower bound l_2 for the second eigenvalue and even a rough value for l_2 is justified since changes in l_2 have little effect on the lower bound calculated from the above expression when l_2 is appreciably greater than μ_2.

Further it is quite reasonable to assume that for flexural vibrations, the second eigenvalue for free vibration of the rotating beam of linearly varying cross-section is a good enough approximation for the lower bound of the second eigenvalue of cantilever beam. Hence calculated value for l_2 is $18.62117 \times 10^4\ \text{sec}^{-2}$.

Thus the frequency for fundamental mode of uncoupled vibration is bounded as

$$4.45707 \times 10^4\ \text{sec}^{-2} < \omega^2 < 4.46058 \times 10^4\ \text{sec}^{-2}$$

The above result shows that the method provided very narrow limits for fundamental frequency of uncoupled vibrations in the presence of base motion.

The fundamental frequency for coupled vibration could not be bounded at present, but the above computation indicates that Rayleigh's quotient obtained from the shape function f_1 is a very close upper bound for the first eigenvalue ω_s^2 for the uncoupled problem. Consequently, it seems reasonable to believe that the method results in a close upper bound for the fundamental frequency for the coupled problem especially if δ is small.

Acknowledgements

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