FREE VIBRATIONS OF BEAMS AND CANTILEVERS WITH ELASTIC RESTRAINTS

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SYNOPSIS

Free undamped flexural vibrations of uniform beams and cantilevers with elastic restraints are studied using Ritz's minimizing method. An approximate mode shape is assumed in terms of suitably selected parameters and by adjusting the relative ratios of these parameters, solutions corresponding to arbitrary elastic restraints applied at the ends of the members are presented. Frequency equations, which otherwise would be in the form of transcendental equations involving trignometric as well as hyperbolic functions, are derived in the form of quadratic or cubic equations. Suitable charts are appended to aid analysis and design.

INTRODUCTION

The governing differential equation for free undamped, flexural vibrations of uniform member of mass m per unit length and uniform flexural rigidity EI, is given by

$$\frac{d^4X}{dx^4} - \lambda^4 X = 0 \qquad \dots (1)$$

in which,

$$\lambda^4 = \frac{mp^2}{EI} \qquad \dots (2)$$

p, being the natural frequency in angular measure; X, is the dynamic deflection mode shape and is of the form

$$X = C_1 \text{ Sin } \lambda x + C_2 \text{Cos } \lambda x + C_3 \text{ Sinh } \lambda x + C_4 \text{ Cosh } \lambda x \quad (3)$$

Four boundary conditions, two at each end of the member, determine these four arbitrary constants and the mode shape is determined to within an arbitrary constant, as follows:

$$\overline{X} = A_1 \left(\text{Sin } \lambda x \frac{C_2}{C_1} \text{Cos } \lambda x + \frac{C_3}{C_1} \text{ Sinh } \lambda x + \frac{C_4}{C_1} \text{ Cosh } \lambda x \right) \quad \dots (4)$$

Only three of the four arbitrary constants are independent and determine the mode shape which is characteristic of the given boundary conditions. The fourth determines its magnitude and has to be determined from the given initial conditions. To calculate the natural frequency only the shape is of concern and is obtained to within an arbitrary constant.

For members with elastic restraints solution of the frequency equation, resulting from the elimination of the four arbitrary constants from the four end conditions of the members, reduces to the solutions of transcendental equations involving trigonometric as well as hyperbolic functions. The solutions will have to be effected graphically, or by trial and error. Since the natural frequency is insensitive to dynamic deflection shape and only depends on the 'overall' or 'average' shape an approximate shape always yields values which are in good agreement with the exact values. Either of the Ritz's methods—the averaging method, or the minimizing one, can be used to formulate the problem. Analysis of beams and cantilevers with elastic restraints at their ends, using these procedures is the main subject matter of the paper. The Ritz minimizing procedure whose usage is more popularly known than that of the averaging method, is used here to derive the expressions.

CANTILEVER WITH ELASTIC MOMENT RESTRAINT AT THE FREE END.

Consider a cantilever of height h, of uniform mass m per unit length, and flexural rigidity EI, as shown in Fig. 1 (a). It is restrained at the top by a moment which is proportional to the rotation thereat. This has been schematically represented in Fig. 1 (a) by a massless spring. Let K be the stiffness of such a restraint. K is the moment per unit rotation, and always opposing the rotation. This may be looked upon as the free body of the column of a portal frame in which the mass of the beam is not considered. Extension of the results obtained for the present case to include the effect of the

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mass will be made. Assume that the dynamic deflection mode shape is of the form

$$X = \frac{a}{2}\left(1-\text{Cos}\ \frac{\pi x}{h}\right)+b\left(1-\text{Cos}\ \frac{\pi x}{2h}\right) \quad (5)$$

This is an approximation to the shape given by Equation (3). The first part of the assumed shape corresponds to the case of infinite restraint as shown in Fig. 1 (c), while as the second part to the case of zero restraint,

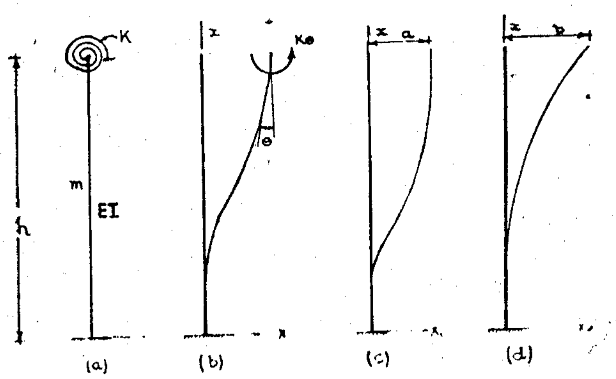

Fig. 1.

or free end, as shown in Fig. 1 (d). Writing Equation (5) in the form analogous to that of Equation (4), namely,

$$\overline{X} = A_1\left[\frac{1}{2}\left(\frac{a}{b}\right)\left(1-\text{Cos}\ \frac{\pi x}{h}\right)+\left(1-\text{Cos}\ \frac{\pi x}{2h}\right)\right] \quad \ldots (6)$$

It can be seen that the ratio $\left(\frac{a}{b}\right)$ determines the shape of the mode, and A_1 the size of it. The shape parameter (a/b) will now be determined as a function of the stiffness of the restraint applied, so that for different stiffnesses, ranging from zero to infinity, this parameter can be adjusted.

The maximum strain energy of the system is given by

$$V = \frac{EI}{2}\int_0^h \left(\frac{d^2X}{dx^2}\right)^2 dx + \tfrac{1}{2} K\,\theta^2 \quad \ldots (7)$$

Evaluating the derivative with respect to x, of X from Eq. (5), at x=h, it can be seen that,

$$\theta = \left.\frac{dX}{dx}\right|_{x=h} = \frac{\pi b}{2h} \quad \ldots (8)$$

Substituting X and θ from Equation (5) and Equation (8) into Equation (7) and evaluating the integrals, the strain energy expression takes the form

$$V = \frac{EI\pi^4}{16h^3}\left[a^2 + \frac{4}{3\pi}ab + \frac{1+\tau}{4}\ b^2\right] \ldots (9)$$

in which τ is the dimensionless ratio of stiffnesses, denoted by

$$\tau = \frac{8}{\pi^2}\cdot\frac{h}{EI}\cdot K \quad \ldots (10)$$

This actually happens to be the approximation of the ratio, $K/\left(\frac{EI}{h}\right)$, in which $\left(\frac{EI}{h}\right)$ represents the flexural stiffness of the cantilever as used in structural statics.

The maximum kinetic energy of the system is given by

$$T = \frac{1}{2}\ mp^2 \int_0^h X^2\, dx \quad \ldots (11)$$

On substituting Equation (5) into this, and evaluating the definite integral, finally,

$$T = \frac{mp^2h}{16}\left[3a^2+8\left(1-\frac{4}{3\pi}\right)ab+8\left(\frac{3}{2}-\frac{4}{\pi}\right)b^2\right] \quad \ldots (12)$$

Equating expressions given by Equations (9) and (12) and denoting,

$$\overline{\lambda} = \left(\frac{\lambda h}{\pi}\right)^4 = \frac{mp^2}{EI}\frac{h^4}{\pi^4} \quad \ldots (13)$$

Rayleigh's quotient takes the form,

$$\overline{\lambda} = \frac{a^2 + \frac{4}{3\pi}ab + \frac{1+\tau}{4}b^2}{3a^2 + 8\left(1-\frac{4}{3\pi}\right)ab+8\left(\frac{3}{2}-\frac{4}{\pi}\right)b^2} \quad (14)$$

Denoting by N and D, the mumerator and denominator, we can write

$$\overline{\lambda} = \frac{N}{D} \quad \ldots (15)$$

Equations

$$\frac{\partial\overline{\lambda}}{\partial a} = 0;\ \frac{\partial\overline{\lambda}}{\partial b} = 0 \quad \ldots (15)$$

determine the conditions to evaluate the parameters a and b. The first of these equations becomes

$$D\ \frac{\partial N}{\partial a} - N\ \frac{\partial D}{\partial a} = 0$$

or

$$D\left(\frac{\partial N}{\partial a}-\frac{N}{D}\ \frac{\partial D}{\partial a}\right)=0$$

Since D cannot be zero and $\frac{N}{D}=\bar{\lambda}$

Equations (15) can be written as

$$\frac{\partial}{\partial a}(N-\bar{\lambda}D)=0$$

$$\frac{\partial}{\partial b}(N-\bar{\lambda}D)=0 \qquad \dots (16)$$

Using for N and D, the numerator and demoninator of Equation (14), these give,

$$\left\{2-6\bar{\lambda}\right\}a+\left\{\frac{4}{3\pi}-8\left(1-\frac{4}{3\pi}\right)\bar{\lambda}\right\}b=0$$

$$\left\{\frac{4}{3\pi}-8\left(1-\frac{4}{3\pi}\right)\bar{\lambda}\right\}a+\left\{\frac{1}{2}(1+\tau)-16\times\left(\frac{3}{2}-\frac{4}{\pi}\right)\bar{\lambda}\right\}b=0 \qquad \dots (17)$$

Therefore, the frequency determinant is as follows :

$$\begin{vmatrix} 2-3\bar{\lambda} & \frac{4}{3\pi}-8\left(1-\frac{4}{3\pi}\right)\bar{\lambda} \\ \frac{4}{3\pi}-8\left(1-\frac{4}{3\pi}\right)\bar{\lambda} & \frac{1}{2}(1+\tau)-16\left(\frac{2}{3}-\frac{4}{\pi}\right)\bar{\lambda} \end{vmatrix}=0 \qquad (18)$$

Expanding the determinant, frequency equation reads as follows :

$$\bar{\lambda}^2-(11.2354+5.3097)\bar{\lambda}+1.4513+1.7699\tau=0 \qquad \dots (19)$$

Frequencies being obtained from this equation, Equations (17) then determine mode shapes. These two equations can be written, in terms of the ratio:

$$\alpha=\left(\frac{a}{b}\right) \qquad \dots (20)$$

From Equations (17)

$$\bar{\lambda}=\frac{\alpha+\frac{2}{3\pi}}{3\alpha+4\left(1-\frac{4}{3\pi}\right)}=\frac{\alpha+0.2122}{3\alpha+2.3024} \qquad \dots (21)$$

$$\alpha=-\frac{\frac{1}{2}(1+\tau)-16\left(\frac{3}{2}-\frac{4}{\pi}\right)\bar{\lambda}}{\frac{4}{3\pi}-8\left(1-\frac{4}{3\pi}\right)\bar{\lambda}} \qquad \dots (22)$$

Between these two, elimination of $\bar{\lambda}$ gives

$$\tau=\frac{6.6632\,\alpha^2+4.2564\,\alpha-0.7627}{3\alpha+2.3024} \qquad \dots (23)$$

From Equation (19) it can be verified that for $\tau=0$ $\bar{\lambda}=0.1307$. This case corresponds to the case of a cantilever. This value of $\bar{\lambda}$ as given by Equation (13), corresponds to $p=3.5681\sqrt{\frac{EI}{mh^4}}$ which is only 1.48% higher than the true value, namely, $3.5159\sqrt{\frac{EI}{mh^4}}$.

The value $\bar{\lambda}=0.1307$ in Equation (21) gives $\alpha=0.1459$. This states the ratios in which the two shapes shown in Figs. 1 (c) and 1 (d) are mixed up. Similarly for the case when $\tau=\infty$ from Equation (19) $\bar{\lambda}=0.3333$ and then from Equation (21) $\alpha=\infty$, that is, $b=0$. This means that the shape is purely as shown in Fig. 1 (c). The value of the frequency will be $p=5.6977\sqrt{\frac{EI}{mL^4}}$ which is of the order only 1.8% higher than the true value $p=5.5932\sqrt{\frac{EI}{mL^4}}$ corresponding to $\bar{\lambda}=0.3211$. The exact frequency equation, as obtained by applying the boundary conditions,

$$\left.\begin{aligned} & X\Big|_{x=0}=0; \\ & \frac{dX}{dx}\Big|_{x=0}=0 \\ & -EI\frac{d^2X}{dx^2}\Big|_{x=h}=K\frac{dX}{dx}\Big|_{x=h} \\ & EI\frac{d^3X}{dx^3}\Big|_{x=0}=0 \end{aligned}\right\} \qquad \dots (24)$$

is as follows :

$$\frac{\text{Cos}\,\lambda L\ \text{Cosh}\,\lambda L+1}{\text{Cos}\,\lambda L\ \text{Sinh}\,\lambda L+\text{Sin}\,\lambda L\ \text{Cosh}\,\lambda L}=\frac{\pi^2}{8}\left(\frac{\tau}{\lambda L}\right) \qquad \dots (25)$$

Equation (19) has taken now the place of Equation (25). The true mode shape governing the frequency equation (25) is quite involved. The mode shape corresponding to equation (25) is as follows :

$$\bar{X}=A_1\{(\text{Sin}\,\lambda x-\text{Sinh}\,\lambda x)+\alpha'(\text{Cos}\,\lambda x-\text{Cosh}\,\lambda x)\} \qquad \dots (26)$$

wherein α' is given by

$$\alpha'=\frac{-(\text{Sin}\,\lambda L+\text{Sinh}\,\lambda L)+\frac{\pi^2}{8}\frac{\tau}{\lambda L}(\text{Cos}\lambda L-\text{Cosh}\,\lambda L)}{(\text{Cos}\,\lambda L+\text{Cosh}\,\lambda L)+\frac{\pi^2}{8}\frac{\tau}{\lambda L}(\text{Sin}\,\lambda L+\text{Sinh}\,\lambda L)} \qquad \dots (27)$$

Equation (6) has here taken the place of Equation (26) and Equation (21) the place of Equation (27). It is seen that Equations (19), (6) and (21) are easier to be handled than their corresponding true expressions—Equations (25), (26) and (27). It is indeed fortunate that frequency is insensitive to these approximations.

Equations (21) and (23) are used to prepare the freqency curve as shown in Fig. 2. To obtain $\bar{\lambda}$ in terms

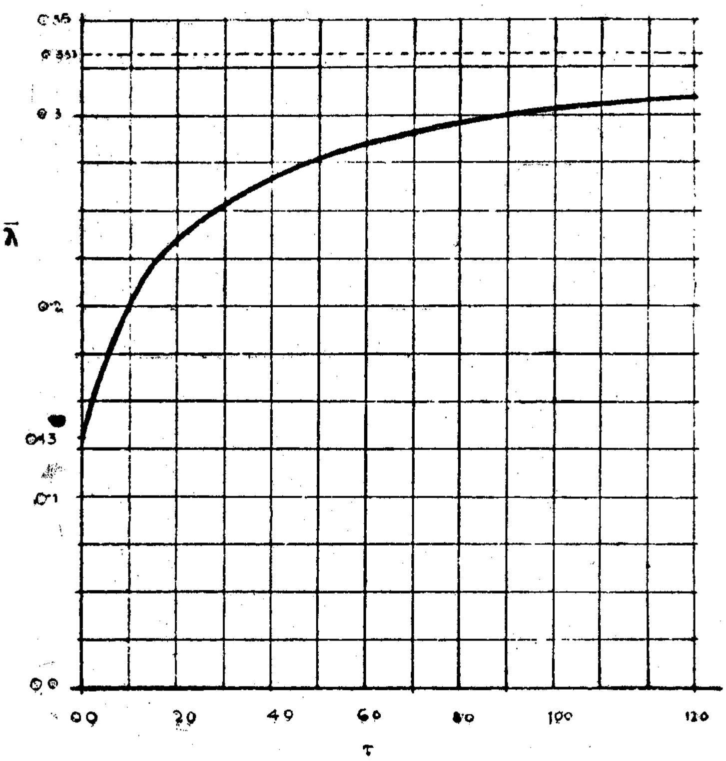

Fig. 2.

of τ, Eq. (19) need be solved. However, Equations (21) and (23) together can yield a set of values to plot $\bar{\lambda}$ Versus τ. Equations (21) and (23) give respectively the plots of $\bar{\lambda}$ Vs α and τ Vs α. From these plots the curve of $\bar{\lambda}$ Vs τ is obtained as shown in Fig. 2. From this figure it is seen the frequency is sensitive to restraints onlyin the beginning upto $\tau=2$ while as it is insensitive beyond $\tau = 10$.

CANTILEVER WITH ELASTIC RESTRAINT AND A MASS CONCENTRATED AT FREE END

The free body of one of the columns of a symmetrical protal frame, shown in Fig. 3 (a) is as shown in Fig. 3 (b).

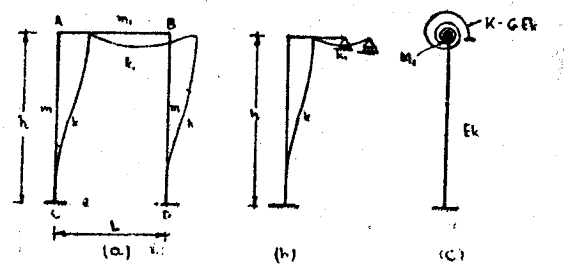

Fig. 3.

The column AB for restraints is equivivalent to a cantilever with moment restraint at top. When sidesway mode of vibrations is considered, one has to consider the kintetic energy as follows :

(a) kinetic energy of the mass of this cantilever assumed uniformly distributed along its length, vibrating horizontally·;

(b) kinetic energy, of the mass of the beam BC, vibrating horizontally, displacement of each of the elemental mass of the beam being equal to the horizontal displace-ment at B of the cantilever, (extensions of the beam being neglected), and

(c) kinetic energy of the mass of the beam BC, asssumed as a distributed mass, vibrating vertically.

Of these three contributions the first two are usually considered and to include the last, the mode shape of the beam BC has to be assumed, consistant with the continuity condition at the joint B. If only the first two are taken into account, the analysis of single bay symmetrical portal frame reduces to that of a cantilever with moment restraint of stiffness,

$$K_1 = 6\,E\,k_1 \qquad \dots (28)$$

and a mass. M_1 concentrated at free end equal to

$$M_1 = m_1 \frac{L}{2} \qquad \dots (29)$$

where m_1 is the mass per unit length of beam. This has been schematically represented in Fig. 3 (c).

To proceed to analyse such a cantilever it will be assumed that the mode shape is still given by Eq. (5). Strain energy expression, given by Equation (9) remains the same, while as to the expression of Kinetic energy, given by Equation (12) the following expression, representing the contribution from the mass M_1 vibrating with sinusoidal displacement having amplitude $X \big|_{x=h}$, namely, (a+b), has to be added:

$$T_1 = \tfrac{1}{2} M_1 (a+b)^2 \qquad \ldots (30)$$

Adding this to the right hand side of Eq. (12) and equating it to the right hand side of Equation (9) one obtains

$$\bar{\lambda} = \frac{a^2 + \frac{4}{3\pi} ab + \frac{1+\tau_1}{4} b^2}{\left(3+\frac{\gamma_1}{8}\right)a^2 + \left\{8\left(1-\frac{4}{3\pi}\right)+\frac{\gamma_1}{4}\right\}ab + \left\{8\left(\frac{3}{2}-\frac{4}{\pi}\right)+\frac{\gamma_1}{8}\right\}b^2} \qquad \ldots (31)$$

in which γ_1 is the ratio of the mass of the beam to that of column,

$$\gamma_1 = \frac{M_1}{mh} = \frac{1}{2} \frac{m_1}{m} \cdot \frac{L}{h} \qquad \ldots (32)$$

and τ_1, the dimensionless ratio of stiffnesses:

$$\tau_1 = \frac{48}{\pi^2} \frac{k_1}{k} \qquad \ldots (33)$$

Equation (31) is the extension of Eq. (14). Proceeding the same way as was adopted in deriving Equations (21) and (22) we obtain instead of them the following equations.

$$\bar{\lambda} = \frac{\alpha + \frac{2}{3\pi}}{\left(3+\frac{\gamma_1}{8}\right)\alpha + 4\left(1-\frac{4}{3\pi}\right)+\frac{\gamma_1}{8}} \qquad \ldots (34)$$

$$\alpha = -\frac{\frac{1}{2}(1+\tau) - \left\{16\left(\frac{3}{2}-\frac{4}{\pi}\right)+\frac{\gamma_1}{4}\right\}\bar{\lambda}}{\frac{4}{3\pi} - \left\{8\left(1-\frac{4}{3\pi}\right)+\frac{\gamma_1}{4}\right\}\bar{\lambda}} \qquad \ldots (35)$$

From these two equations $\bar{\lambda}$ and α can be obtained for given values of τ_1 and γ_1 either by trial and error, or by solving the resulting quadratic by elimination of α between these two equations. The quadratic would be the same form as equation (19).

BEAMS WITH ELASTIC RESTRAINTS.

Analysis of the free flexural vibrations will now be carried to the case of beams with uniformly distributed mass and flexural rigidity. The ends of the beam are held against relative displacement, but are elastically restrained against rotations, as shown schematically in Fig. 4 (a). This may be looked upon as the freebody of the vertically vibrating beam AB of a portal frame shown in Fig. 5, in which the columns have different stifnesses, In analysing the beam AB of Fig. 4 (a) the

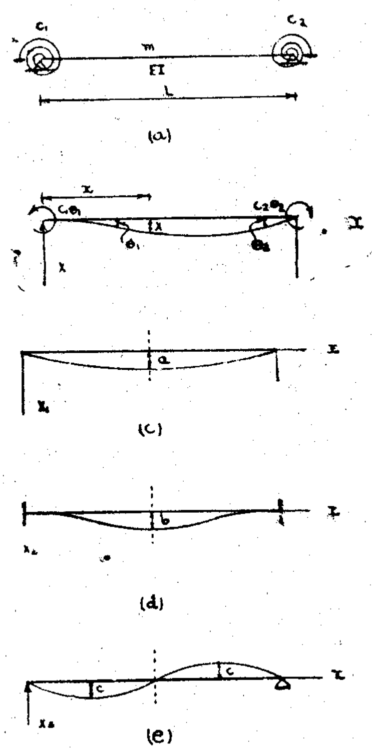

Fig. 4.

springs will be assumed massless. It differs from being the complete dynamic analogue of the beam AB of the portal frame of Fig. 5, in as much as the Kinetic energy

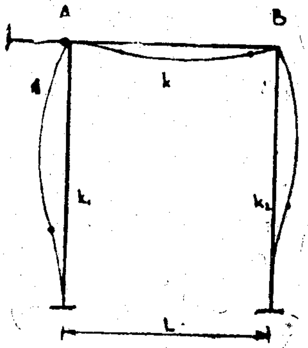

Fig. 5.

of the column masses vibrating horizontally is excluded. The results obtained for the beam AB hold for the beam

AB of the portal frame in which the masses of the columns are either too small or are prevented from vibrating laterally.

For the beam AB of Fig. 4 (a) assume the dynamic deflection mode shape as

$$X = a \sin\frac{\pi x}{L} + \frac{b}{2}\left(1-\cos\frac{2\pi x}{L}\right) + c \sin\frac{2\pi x}{L} \quad (36)$$

Each of the shapes are shown in Figs. 4 (c), 4 (d) and 4 (e). The shape function will be

$$\bar{X} = A_1\left\{\sin\frac{\pi x}{L} + \frac{\alpha_1}{2}\left(1-\cos\frac{2\pi x}{2}\right) + \alpha_2 \sin\frac{2\pi x}{L}\right\} \quad \dots (37)$$

where

$$\alpha_1 = \left(\frac{b}{a}\right); \ \alpha_2 = \left(\frac{c}{a}\right) \quad \dots (38)$$

and A_1 is the size parameter. As in the previous case, here too α_1 and α_2 will be determined in terms of the stifnesses C_1 and C_2.

Maximum strain-energy will be given by

$$V = \frac{EI}{2}\int_0^L \left(\frac{d^2X}{dx^2}\right)^2 dx + \tfrac{1}{2}C_1\theta_1^2 + \tfrac{1}{2}C_2\theta_2^2 \quad (39)$$

where the second and third terms on the right hand side represent the energy stored in restraints at the supports. Evaluating the drivative with respect to x of X as stated by Equation (36) at x = 0, and x = L, one obtains

$$\theta_1 = \frac{\pi}{L}(a+2c); \ \theta_2 = \frac{\pi}{L}(-a+2c) \quad \dots (40)$$

Substituting Equation (36) and Equation (40) in Equation (39) and evaluating the definite integrals, one arrives at

$$V = \frac{EI\pi^4}{4L^3}\left[a^2+4b^2+16c^2+\frac{16}{3\pi}ab+(a^2+4c^2)(\tau_1+\tau_2)+4ac(\tau_1-\tau_2)\right] \quad \dots (41)$$

in which τ_1 and τ_2 are the relative stiffness ratios

$$\tau_1 = \frac{C_1}{\left(\frac{\pi^2 EI}{2L}\right)}; \ \tau_2 = \frac{C_2}{\left(\frac{\pi^2 EI}{2L}\right)} \quad \dots (42)$$

The factor $\frac{\pi^2 EI}{2L}$ is Fourier approximation of the exact value $\frac{4EI}{L}$, representing the flexural stiffness of beam without vibrations. Substituting for x, its value given by Equations (36) into the right-hand side of Equation (11) and evaluating the integrals, expression for maximum Kinetic Energy becomes

$$T = \frac{mp^2L}{4}\left\{a^2 + \tfrac{3}{4}b^2 + c^2 + \frac{16}{3\pi}ab\right\} \quad \dots (43)$$

Equating Equation (41) to Equation (43) and denoting

$$\bar{\lambda} = \left(\frac{\lambda L}{\pi}\right)^4 = \frac{mp^2}{EI}\cdot\frac{L^4}{\pi^4} \quad \dots (44)$$

one abtains

$$\bar{n} = \frac{a^2+4b^2+16c^2+\frac{16}{3\pi}ab+(a^2+4c^2)(\tau_1+\tau_2)+4ac(\tau_1-\tau_2)}{a^2+\frac{3}{4}b^2+c^2+\frac{16}{3\pi}ab} \quad \dots (45)$$

Proceeding in the same way as was adopted in deriving Equations (16),

$$\left.\begin{aligned} \frac{\partial}{\partial a}(N-\bar{\lambda}D) &= 0 \\ \frac{\partial}{\partial b}(N-\bar{\lambda}D) &= 0 \\ \frac{\partial}{\partial c}(N-\bar{\lambda}D) &= 0 \end{aligned}\right\} \quad \dots (46)$$

Where N and D respectively denote the numerator and denominator of Equation (45). These equations give the governing equations of mode shapes. These are as follows :

$$\left.\begin{aligned} \left\{1+(\tau_1+\tau_2)-\bar{\lambda}\right\}a + \frac{8}{3\pi}(1-\bar{\lambda})b + 2(\tau_1-\tau_2)c &= 0 \\ \frac{8}{3\pi}(1-)\bar{\lambda}a + (4-\tfrac{3}{4}\lambda)b &= 0 \\ 2(\tau_1-\tau_2)a + \{16+4(\tau_1+\tau)-\bar{\lambda}\}c &= 0 \end{aligned}\right\} \quad (47)$$

The condition that the determinant of these equations should vanish yields the frequency equation. Alternatively by a systematic elimination of a, b and c the same equation can be derived. From second and third equations of (47), ratios (b/a) can be obtained. Using the notation stated by (38)

$$\alpha_1 = \frac{\frac{8}{3\pi}(1-\bar{\lambda})}{\frac{3}{4}\bar{\lambda}-4} \quad \dots (48)$$

$$a_2 = \frac{2(\tau_1 - \tau_2)}{\bar{\lambda} - 4(\tau_1 + \tau_2) - 16} \quad \ldots \quad (49)$$

Dividing the first of Eqs. (47) by a throughout and using therein Eqs. (48) and (49) and simplifying the frequency equation will be obtained as

$$\begin{aligned} &0.0295\bar{\lambda}^3 - \bar{\lambda}^2[0.868(\tau_1+\tau_2) + 3.7810] \\ &+ \bar{\lambda}[29.2360(\tau_1+\tau_2) + 12\,\tau_1\tau_2 + 56.2240] \\ &- 64\,\tau_1\tau_2 - 77.1180(\tau_1+\tau_2) - 52.4720 = 0 \quad (50) \end{aligned}$$

$\bar{\lambda}$ being determined by this equation Equations (48) and Equations (49) will then give the mode shape parameters a_1 and a_2.

It is seen that Equation (50) is symmetrical with respect to τ_1 and τ_2. For the case of fixed beam $\tau_1 = \tau_2 = \infty$, Equation (50) gives

$\bar{\lambda} = \frac{16}{3}$. Using Eq.(44), $p = 22.7985\sqrt{\frac{EI}{mL^4}}$ which is about 1.5% higher than the true value of $22.36\sqrt{\frac{EI}{mL^4}}$ For this case $a_2 = 0$ and $a = \infty$ that is, $a = 0$, as given by Eqs. (48) and (49).

Preparation of frequency charts in this case is more

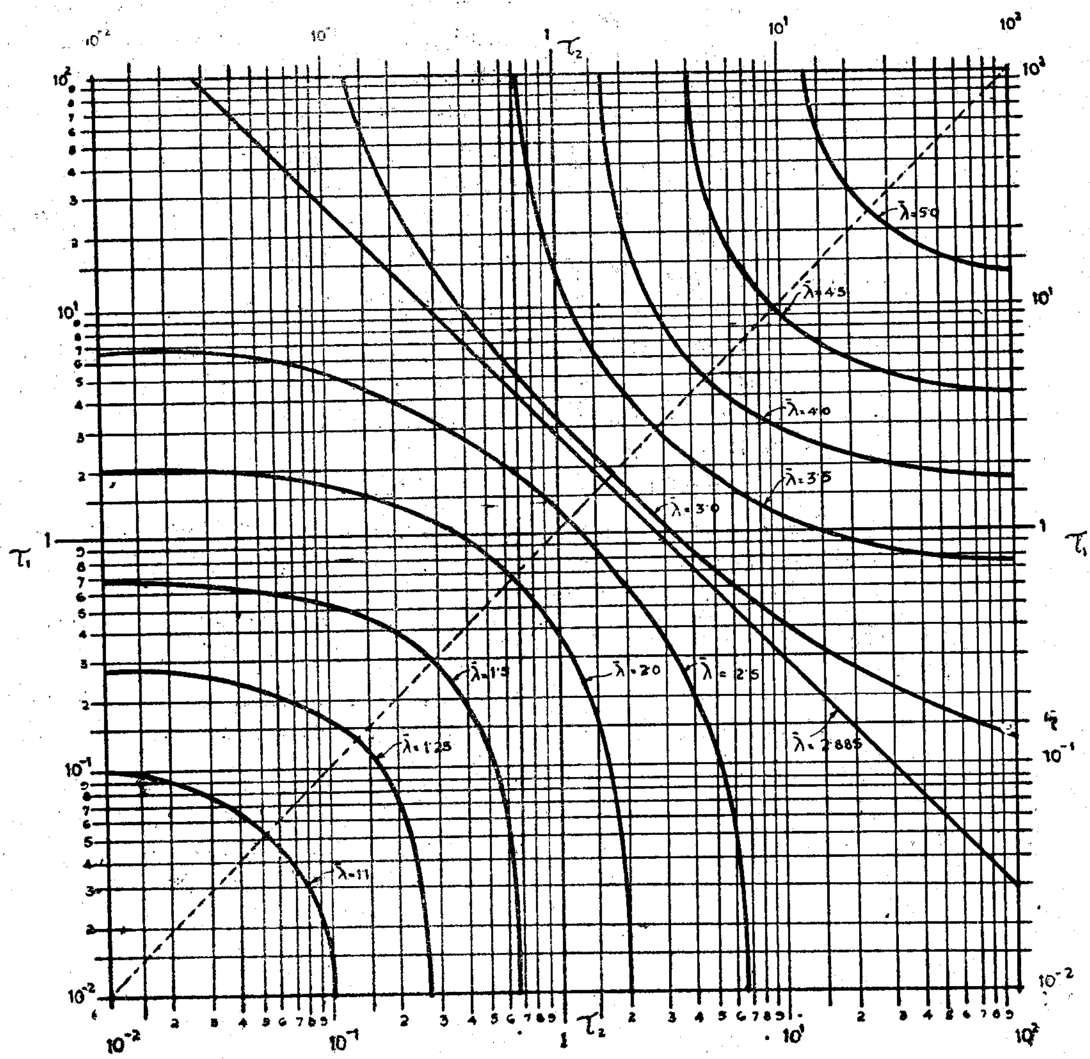

Fig. 6.

involved than in the previous case of cantilever in which only one parameter α was involved. In this case Equation (50) has been solved for τ_2 in terms of τ_1 and $\bar{\lambda}$, and for different constant values of $\bar{\lambda}$ τ_2 has been solved in terms of τ_1. For example for $\bar{\lambda} = 2$,

Equation (50) reduces to

$$\tau_1+\tau_2+1.8080\,\tau_1\,\tau_2-2.0380 = 0 \quad \dots \quad (51)$$

This is symmetrical with respect to the diagonal line $\tau_2=\tau_1$ (Fig. 6). Any set of values for τ_1 and τ_2 that satisfies this equation gives frequency $\bar{\lambda} = 2$. If $\tau_1=\tau_2$, that is, if the end restraints are identical, then $\tau_1=\tau_2=0.644$. Also if one of the ends is hinged, then the other should have the restraint of stiffness ratio equal to 2.038, as obtained by putting either of τ_1 or τ_2, equal to zero. Equations similar to Equation (51) have been determined for different assigned values for $\bar{\lambda}$ rangingfrom $\bar{\lambda} = 1$ to $\bar{\lambda} = \frac{16}{3}$, respectively the lower and the upper bounds for the frequency.

Curves given in Fig. (6) can be used to try the various stiffness ratios to produce a desired frequency, or conversely, given the stiffness ratios of the restraints to calculate the frequency. Since the curves are on logarithmic scale interpolation may cause errors. Equations (48), (49) and (50) can always be used to rectify the values picked up from the curves. For example suppose $\tau_1=\tau_2=1$. From the curves $\bar{\lambda} = 2.4$. To correct it for inaccuracies involved in interpolation, Newton's formula can be applied to Eq. (50). It was found that the first correction was -0.02 giving $\bar{\lambda} = 2.38$. A second correction gave $\bar{\lambda} = 2.3802$.

Mode shape can be determined from Equations (48) and Equations (49) when once the value of $\bar{\lambda}$ is known. Suppose $\bar{\lambda}=2.5$. From the frequency chart, $\tau_1=2$, $\tau_2=0.6$ is one of the admissible sets. Eqbations (48) and (49) for these values of $\bar{\lambda}$, τ_1 and τ_2 give $\alpha_1 = 0.5992$ and $\alpha_2 = -0.1171$. Therefore, Eq. (37) becomes

$$\bar{X}=A_1\left\{\text{Sin}\,\frac{\pi x}{L}+0.2996\left(1-\text{Cos}\,\frac{2\pi x}{L}\right)-0.1171\,\text{Sin}\,\frac{2\pi x}{L}\right\} \quad \dots \quad (52)$$

This shape function corresponds to points of contrflexures at $x = 0.22\,L$ and $x = 0.86\,L$.

CONCLUDING REMARKS.

By assuming an approximate shape for the dynamic deflection mode in terms of some suitably selected parameters solution for natural frequencies of beams and cantilevers with elastic restraints can be effected with a greater ease. The errors admitted in the frequencies so obtained are of permissible order. Solution of frequency equations which are transcendental in nature involving trigonometric and hyperbolic functions are replaced by algebraic equations. It is easier to vary the parameters and prepare the frequency charts.

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