Paper No. 183, Bulletin, I.S.E.T. Vol. 15, No. 3, September 1978, pp. 39–49

DYNAMIC ANALYSIS OF MULTI-STOREY FRAMES
(A comparison of various methods)

by Dr. R. RADHAKRISHNAN* AND Dr. N. SUBRAMANIAN**

INTRODUCTION

Natural Frequency and Mode Shape

Classical Approach

The modal shapes, frequencies and the modal damping values define the dynamic characteristics of a structure. The free undamped vibration equation of motion is the basic one for solving these parameters and is given by (1)

$$[M]\{\ddot{x}\}+[k]\{x\}=0 \qquad \ldots(1)$$

The natural frequencies are exactly determinable from the frequency determinant:

$$[k]-p^2[M]=0 \qquad \ldots(2)$$

The modal patterns can be obtained from the equation

$$([k]-p^2[M])\{x\}=0 \qquad \ldots(3)$$

where $\{x\}$ is the vector of amplitudes.

A second method is to convert the equation (1) into the mathematical eigen value problem. Defining $[D]$ as the dynamic matrix$=[M]^{-1}[k]$, the equation (1) reduces to the form

$$[D]\{x\}=p^2\{x\} \qquad \ldots(4)$$

The equation (4) is in the algebraic form of the eigen value problem where p^2 is the eigen value and $\{x\}$ is the eigen vector. The solution will give 'n' eigen values which will represent the squares of the 'n' natural frequencies of vibration and 'n' eigen vectors which will be the 'n' sets of normalised mode shapes.

Although equations (2) and (3) have a compact matrix form, the task of finding the roots of an n^{th} order determinant becomes laborious and an electronic computer is required. The solution of equation (4) also involves mathematical techniques and the use of an electronic computer.

With frequency equation approach, it is necessary to solve the complete eigen value problem even if only a few of the modes are desired. In practice, a knowledge of the lowest or a few of the lowest frequencies and associated mode shapes are generally sufficient for the solution of many engineering problems. Approximate and numerical procedures can be used in place of the frequency determinant to obtain the requisite information only (2, 3).

*Assistant Professor, Department of Civil Engineering, Indian Institute of Technology, Madras—600036.
**Project Associate, Ocean Engineering Centre, Indian Institute of Technology, Madras—600 036.

Natural Period Patterns

John A. Blume's study on cantilever type buildings (4) revealed that for structures with very flexible beams, the period ratios T_1/T_2 and T_1/T_3 approaches the classical ratios of 6.27 and 17.6 for uniform cantilever bars in flexure. T_1, T_2 and T_3 are the natural period of vibration of fundamental, 2nd and 3rd modes respectively. When the beams approached heavy rigidity, the period ratios approached the classical shear bar values of 3 and 5. Housner and Brady's study on a variety of modern steel frame buildings in Southern California (5) revealed that the fundamental periods are best described by an equation of the form: $T=a\sqrt{n}-b$ where 'a' and 'b' are the coefficients defined from measured values of periods and 'n' is the total number of stories.

CODE PROVISIONS

The codes for earthquake resistant structures (6) prescribe the fundamental periods and the provisions in the codes of some of the countries are given in Table 1.

TABLE I. CODE RULES FOR FUNDAMENTAL PERIODS

S.No.	Country	Formula	Notations	Remarks
1.	Bulgaria	$0.19n$	n=Number of Stories H=height (m) of building D=depth (m) in direction of Seismic force	Frames without braces
		$0.09H/\sqrt{D}$		With braces
2.	India	$0.1n$	—do—	Moment Resistant frames
		$0.09H/\sqrt{D}$	—do—	Others
3.	Canada	$0.1n$	H and D in feet	Moment resistant frames
		$0.05H/\sqrt{D}$		Others
4.	Iran	$0.09H/\sqrt{D}$	—do—	All frames
5.	NewZealand	$0.32\sqrt{D}\pm20\%$	—do—	Code limitation
		$0.1n$	—do—	Frames
		$0.05H/\sqrt{D}$	—do—	Frame cum shear wall

S.No.	Country	Formula	Notations	Remarks
6.	Peru	$0.1n$		Frame
		$0.09H/\sqrt{D}$		Frame and few shear walls
		$0.07H/\sqrt{D}$	—do—	Frame with some shear walls
		$0.05H/\sqrt{D}$		Frame with many shear walls
7.	Phillipines	$0.1n$		Moment resistant frames
		$0.05H/\sqrt{D}$	—do—	Normal buildings
8.	Spain	$2\pi\sqrt{\frac{\Sigma m^3u^3}{\Sigma m^2u^2}}$	m=lumped masses u=flexibility	all frames
9.	U.S.A.	$0.1n$	—	Moment resistant frames

MODE SUPERPOSITION OF SPECTRUM RESPONSE

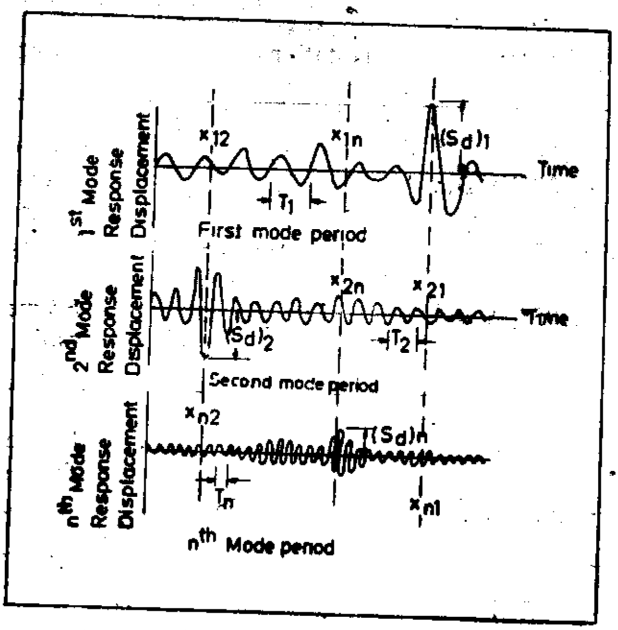
Housner's average acceleration spectrum and average velocity spectrum for earthquakes (7) have been adopted by the Indian Code (8). The maximum response values of relative displacement, relative velocity and absolute acceleration determined for a single degree of freedom system by the method of Duhamel integral (9) during the time history of earthquake are denoted by S_d, S_v and S_a. Plots of S_d, S_v and S_a against undamped natural periods for various fractions of critical damping provide the earthquake response spectra for relative displacements, relative velocity and absolute acceleration respectively. Hudson (10) has demonstrated that if damping is small, approximations resulting in the following simple relationships can be plausibly made, although no rigourous demonstration of the errors involved seems possible.

$$S_d = \frac{1}{p_d} S_v \qquad \ldots(5)$$

$$S_a = p_d S_v \qquad \ldots(6)$$

where p_d is the damped natural frequency of vibration.

Fig. 1 would demonstrate that the S_d values and consequently the S_v and S_a values of the different modes do not occur simultaneously. The values read directly from the



$(S_d)_1$ Spectral max for mode 1
$(S_d)_2$,, ,, ,, 2
$(S_d)_3$,, ,, ,, n
$\times 1n$: Disp. of mode 1 at time inst. when mode n disp. is max.
$\times n1$: Disp. of mode n at time inst. when model disp. is max.

Fig. 1. Mode superposition in multi degree of freedom systems.

Housner's response spectra provide only values of S_d, S_v and S_a for each mode. The total effect of 'n' modes considered at the time instant when $(S_d)_1$ for mode 1 occurs is $\sum_{n=1}^{n} x_{n1}$ and not $\sum_{n=1}^{n} (S_d)_n$, the notations of which have been explained in Fig. 1. The total effect at the time when $(S_d)_i$ for mode i occures is similarly $\sum_{n=1}^{n} x_{ni}$ and is less than $\sum_{n=1}^{n} (S_d)_n$.

For practical designs, a knowledge as to the number of modes that are significant and a technique for predicting from the response spectra the total response values of all modes at the same time instant are required. The modes which are significant are fairly determinable from the product of equivalent single degree mass of each mode and the corresponding S_a(11). On the basis of his study on Alexander Building Clough (12) concluded that on the average, the second and third modes of vibration add about 15 percent to the shear forces developed in the structure in the first mode of vibration and that about 36 percent of the second-mode spectral response value and 41 percent of the third-mode spectral response value is present in the maximum total shear force.

Biot suggested that a suitable method for mode combination is to take the sum of the absolute values of the individual modes (9). This would obviously give the worst possible combination and would thus set an upper bound to the response. Fung and Barton showed for pulse type excitations that algebraic sum of the individual modes will give a better value than absolute sum, for higher frequency responses (9). For earthquake type excitations, they however prefer the absolute sum.

Rosenblueth suggested from a probabilistic approach that the mode spectral values may be combined as the square root of the sum of the squares (RSS) which is also known in earthquake literature as Root mean square value (RMS) (9). Jennings suggested a refinement involving the average of the absolute sum and the square root of the sum of the squares (13). Merchant and Hudson suggested a weighted average of absolute sum and RSS of the form

$$\frac{\text{RSS}+m\,(\text{abs. Sum})}{m+1} \qquad \ldots(7)$$

Where '*m*' is a weighting factor by adjusting which the maximum non-conservative deviation could be limited to any desired level at the expense of an increased number of positive deviations (13). Husid and Ronnberg (14, 15) demonstrated that for Chimney-like structures, the RSS method used in several earthquake design codes gives a poor estimate of the exact response, mostly on the lower side. The Indian standards Institution in their code IS 1893–1975 have adopted the above idea of Merchant and Hudson to suggest a weighted average of the form:

$$r\,(\text{RSS})+(1-r)\,(\text{Abs. sum}) \qquad \ldots(8)$$

in which *r* is a weighting factor having values:

0.40	for building heights less than	20 m.
0.60	,,	=40 m.
0.80	,,	=60 m.
1.00	,, greater than	90 m.

EARTHQUAKE RESISTANT STANDARDS

Codes of most of the countries lay down a pseudo dynamic method or modal analysis for the design of structures against earthquakes depending upon the height and importance of the structure. The Indian Standard IS : 1893–1975 lays down a pseudo-dynamic method of seismic coefficients for buildings of heights not exceeding 40 m to 90 m and a rigorous dynamic design for buildings whose height exceeds 90 m. The standards of Aseismic Civil Engineering Construction in Japan lays down a seismic coefficient method with a provision that other superior methods are also allowed. It is the normal practice in Japan to adopt the response spectrum method for lowrise buildings (upto about 10 storeys) and to adopt a rigorous dynamic analysis for higher structures. The uniform Building Code 1970 Vol. I of U.S.A. lays down a pseudo dynamic system of design taking into account the principles of dynamic response and also specify factors conecting period and other parameters of the Building in determining the lateral forces. The provisions of some of the other countries are indicated below in Table 2, with an asterisk.

The methods of mode superposition for analysis by response spectra, as laid down in some of the codes are listed in Table 3.

DISCUSSIONS ON CODE PROVISIONS

For dynamic analysis of structures, the parameters of period of free vibration, mode shape and damping values of the structure are most important. While the period and mode shapes can be determined in a rigorous manner by the aid of an electronic digital computer, there should be simpler methods available for determining them at

TABLE 2. CODE PROVISIONS FOR EARTHQUAKE RESPONSE ANALYSIS

S. No.	Country	Static	Pseudo dynamic	Modal Analysis by Response Spectrum	Rigorous Dynamic Analysis
1.	Bulgaria		×		
2.	Canada		×		
3.	Cuba	×	×		
4.	Elsalvador	×			×
5.	France		×		
6.	Mexico	×		×	×
7.	NewZealand		×		
8.	Peru		×	×	
9.	Phillipines		×		
10.	Rumania		×	×	
11.	Spain		×		
12.	Turkey	×			

TABLE 3. METHODS OF MODE SUPERPOSITIONS IN VARIOUS CODES

Sl. No.	Country	Method	Notations	Remarks
1.	India	RMS	—	Same as RSS
2.	Elsalvador	RSS	—	—
3.	Peru	RSS		
4.	Rumania	$\sqrt{N_a^2+0.5\,(N_b^2+N_c^2)}$	N_a=Mode-1 Response N_b=Mode-2 Response N_c=Mode-3 Response	

least for preliminary designs. Apart from the limitation that higher periods cannot be obtained from the codes rules, the empirical formulae laid down in the codes of various countries for determining fundamental periods will represent only generally applicable values and thus may be poor approximation for particular stiffness-mass patterns. Simple methods are also not available for determining the mode shapes to a good degree of accuracy.

COMPUTER ORIENTED PROCEDURE

In equation (4), D is known and $\{x\}$ and p^2 are unknown. The matrix $[D]$ is then defined in the following form as

$$[D][B]=[B][A] \qquad \ldots(8)$$

where $[B]$ is an orthogonal square matrix and $[A]$ is a diagonal matrix. If the above equation is possible, $[A]$ will give the 'n' values of p^2 and $[B]$ the 'n' sets of vectors of $\{x\}$, which are the desired solutions.

For determining the Eigen values and Eigen vectors of a building frame, the dynamic matrix $[D]$ is first prepared. Thereafter, the procedure is to find an n^{th} order orthogonal matrix $[B]$ with element b_{ij}, such that the transform of $[D]$ by $[B]$ is the diagonal matrix $[A]$.

i.e.,

$$[B]^T[D][B]=[A] \qquad \ldots(9)$$

As the matrix $[B]$ is orthogonal, premultiplication of both sides of expression (9) with $[B]$ would result in expression (8) and hence this procedure. For the transformation of the dynamic matrix, into a diagonal matrix, the Jacobi method is used in the program (9). Side by side with the diagonalisation of the matrix $[D]$ by iteration, the orthogonal matrix $[B]$ is also generated under each step. Each step corrects the values of elements assigned in previous steps, until the diagonalisation is completed and the orthogonal matrix is simultaneously formed, according to the convergence criterion defined.

Based on the above procedure a computer program was developed in FORTRAN IV Language to run on IBM 370/155 machine. The input data consist of the number of storeys, storey stiffnesses and floor masses. The program generates the dynamic matrix and as explained above, the dynamic matrix is diagonalised and the eigen values and eigen vectors are determined.

SIMPLIFIED EQUATIONS

It was found that most real buildings either lie in the shear zone or fall close to shear zone of frequency pattern. Hence, about one hundred and fifty-five shear frames of 5 to 40 storeys, with various column stiffness patterns were analysed using the computer program. From the results, the fundamental period, T, for uniform stiffness-uniform mass shear frames is given by following expression

$$T^2=\frac{M}{K}\,(16n^2+16n+7.3) \qquad \ldots(10)$$

Hence from the known values of K and M, the fundamental period of a uniform shear building can be determined to a high degree of accuracy from the above expression by very simple calculations (16).

The frequency Ratios of the various modes of the Uniform shear building have fixed values as indicated in Table 4. Hence from the fundamental period calculated as above, the period of any mode can be obtained to a very high degree of accuracy by making use of this frequency ratio. Tables have also been worked out for directly reading the mode shapes for the frames having 5 to 40 storeys and are not shown here for brevity.

TABLE 4. FREQUENCY RATIOS FOR UNIFORM SHEAR BUILDINGS

n=5	n=10	n=15
1.00	1.00	1.00
2.92	2.98	2.99
4.60	4.89	4.95
5.91	6.69	6.86
6.74	8.34	8.69
	9.80	10.44
	11.06	12.09
	12.06	13.60
	12.79	14.98
	13.23	16.20
		17.26
		18.14
		18.84
		19.34
		19.64

For the taper stiffness frames, (with their stiffnesses increasing from top to bottom at a uniform rate per floor) a moment of inertia pattern parameter, f, is defined as

$$f=1+(MI\text{ variation rate})/(MI\text{ of top most storey column}) \quad \ldots(11)$$

for these frames, the fundamental period is given by the equation,

$$T^2=\frac{M}{K}(an^3+bn^2+cn+d) \quad \ldots(12)$$

where the values of a, b, c and d are given in Table 5. For other values of f, the values can be interpolated. The frequency ratios and mode shapes have been worked in tabular form for these frames also and are not included here for brevity.

TABLE 5. COEFFICIENTS FOR TAPER SHEAR BUILDINGS

Sl. No.	Fundamental Frequency in Sec.	$T^2=\frac{M}{K}(an^3+bn^2+cn+d)$			
	f	a	b	c	d
1.	1	−0.0000579	16.003805	15.93447	7.540182
2.	$1\frac{1}{8}$	0.0007248	13.270240	22.88256	3.324875
3.	$1\frac{1}{5}$	0.0023351	13.221467	22.29090	3.963298
4.	$1\frac{1}{8}$	0.0038194	13.271370	20.11338	6.089831
5.	$1\frac{1}{2}$	0.0025998	13.432256	18.75490	7.288665
6.	$1\frac{1}{2}$	0.0012878	13.551899	18.63797	7.287244

COMPARISON

For comparison an uniform stiffness-uniform mass 5-storeyed building frame with the following values of mass and stiffness are taken. M_i (i=1 to 5)=86.83 kg s^2 cm^{-1}, k_i (i=1 to 5)=41, 943 kg cm^{-1}, H=12 m, D=9 m. The values found out by using various formulae are given in Table 6. From this table it is found that the proposed equation gives more reliable values than the formulae suggested by the standard codes.

From Table 4, the second, third, fourth and fith mode frequencies are found out as 0.3441 s, 0.218 s, 0.170 s and 0.149 s respectively. These values tally exactly with the values calculated by a rigorous computer analysis, namely T_2=0.3441, T_3=0.2183, T_4=0.1699, T_5=0.1490. Formulae are not available in standard codes for finding the second, third, fourth and fifth mode frequencies. Moreover, they also don't take into account the moment of inertia variation of the columns. Hence the proposed formulae are more useful for any general frame and could be included in the Indian code for obtaining more reasonable and accurate results.

SUMMARY AND CONCLUSION

A brief review is made on the various methods used for predicting natural period patterns and mode superposition of spectrum response. The earthquake resistant standards of various countries are discussed. A rigorous computer method developed

TABLE 4. COMPARISON OF FORMULAE

Sl. No.	Formula	Fundamental Frequency in Sec
1.	Bulgaria code	
	(i) Frames without braces	0.95
	(ii) With braces	0.36
2.	Indian code	
	(i) Moment resistant frames	0.5
	(iii Other frames	0.36
3.	Canada	
	(i) Moment resistant frames	0.5
	(ii) Other frames	0.362
4.	Iran	0.652
5.	NewZealand	
	(i) Frames	0.5
	(ii) Frame cum shear wall	0.362
6.	Peru	
	(i) Frame	0.5
	(ii) Frame with few shear walls	0.652
	(iii) Frame with some shear walls	0.507
	(iv) Frame with many shear walls	0.362
7.	Phillipines	
	(i) Moment Resistant frames	0.5
	(ii) Normal buildings	0.362
8.	Spain	0.286
9.	U.S.A.	0.5
10.	Proposed equation	1.0044
11.	Computer solution	1.0044

is discussed. From the one hundred and fifty five models analysed a simplified equation is proposed. It is shown this equation gives more reliable values and hence is suggested for the Indian Code. Tables have been provided to find all mode frequency response of any general shear frames.

REFERENCES

1. Timoshenko, S.P, Vibration Problems in Engineering (3rd ed), D. Van Nostrand Company, Inc., Princeton, New Jersey, 1955.
2. Hurty, W.C. and Rubinstein, M.F., Dynamics of Structures, Prentice-Hall of India. Pvt. Ltd., New Delhi, 1967.

3. Clough, R.W., and Penzien, J. Dynamics of Structures, McGraw Hill Book Co., New York. 1975.
4. Blume, J.A., 'Structural Dynamics of Cantilever type Buildings', Proceedings of the Forth World conference on Earthquake Engineering, 1969, Chile, Vol. II, A-3, pp. 1–13.
5. Housner, G.W., and Brady, A.G., 'Natural Periods of Vibration of Buildings', Journal of the Engineering Mechanics Div., ASCE, Vol. 89, No. EM 4, Proc. paper 3613, Aug. 1963, pp. 31–65.
6. ——— Earthquake Resistant Regulations—A world list 1970. International Association for Earthquake Engineering, Tokyo, 1970.
7. Housner G.W., 'Behaviour of Structures During Earthquakes' Journal of the Engineering Mechanics Div., ASCE, Vol. 85, No. EM 4, Proc. Paper 2220, Oct. 1959, pp. 109–128.
8. — — — Indian Standard Criteria for Earthquake Resistant Design of Structures, Third Revision, IS: 1893–1975, Indian Standards Institution, New Delhi, 1975.
9. Joseph, M.G., Dynamic Effects of Earthquake and wind on Multistoried Buildings, Thesis submitted for the award of Ph.D. Degree to the Indian Institute of Technology, Madras, 1976.
10. Hudson, D.E., 'A comparison of Theoretical and Experimental Determinations of Building Response to Earthquake', Proc. of the Second World Conference on Earthquake Engineering, Tokyo, Japan, 1960. pp. 1105–1118.
11. Joseph, M.G., 'Equivalent single Degree Structure Analogy in Dynamic Analysis of Tall Buildings against lateral Forces and a Physical Concept Approach for Building Frameworks. The Bridge and Structural Engineer, Journal of ING/IABSE, Vol. 5. No. 2, Dec. 1975 pp. 47–75.
12. Clough. R.W., 'On the Importance of Higher Modes of Vibration in the Earthquake Response of a Tall Building', Bulletin of the Seism. Soc. of America, Vol. 45, No. 4, Aug. 1955.
13. Merchant, H C., and Hudson, D.E., 'Mode Superposition in Multi-degree of Freedom systems Using Earthquake Response Spectrum Data', Bulletin of the seism. Soc. of America, Vol. 52, No. 2, April 1962, pp. 405–416.
14. Husid, R., and Ronnberg, C., "Exact Response and Model Superposition Techniques of Bending Structures with Earthquake Excitation', Symp. on Earthquake Engineering, University of Roorkee,' Nov. 1970, pp. 254–261.
15. Husid, R. and Ronnberg, C., 'The Estimation of the Exact Response of Chimneys with Earthquake Excitation', Proc. of the Third Earthquake Engineering Symp., Nov, 1970, III–11, pp. 389-396.
16. Joseph, M.G., and Radhakrishnan. R., 'Natural patterns of vibration of building frames', Journal of Sound and Vibration, Vol. 46, No. 1. 1976, pp. 15–32.