Paper No. 90. Bulletin of the Indian Society of Earthquake Technology, Vol.VII, No. 2, June 1970, pp 89–95

BEAM VIBRATIONS WITH CENTRALLY ATTACHED MASS

Y. C. Das*, Member and R. S. Deshmukh**

INTRODUCTION

Frequencies of beams with attached mass are quite often needed to study the behaviour of structural component in a structural system. Vibrations of beams carrying mass having moment of inertia have been studied in recent years by several authors (1, 2, 3). In all these cases, effects of rotary inertia and shear deformation of the beam on the frequencies of the beam-mass system are neglected. It is well known that classical Euler-Bernoulli theory is inadequate for the study of higher modes of the beam as well for the modes of beam for which cross-sectional dimensions are not small compared to their length between nodal sections.

The aim of this paper is to study the vibrations of a beam with central attached mass having a finite moment of inertia, including the effects of rotary inertia and shear deformation of the beam. Due to symmetry of boundary conditions, and the central location of the mass, two types of modes, viz. symmetric and asymmetric, exist. By using this property, only half of the beam is considered to derive the frequency equations. In case of symmetric modes, there is only translatory displacement of the attached mass in transverse direction, and in case of asymmetric modes, there is only rotation of the attached mass. These properties are used in setting up the proper conditions at the centre of the beam. The general characteristic equations of the beam mass systems are highly transcendental in nature and are solved by the help of a digital computer to get the first two frequencies.

This problem has a lot of bearing on practical situations A machine resting on a deep beam or the wings-par section of an aircraft may be idealised to one of the cases in this paper.

ANALYSIS OF THE PROBLEM

The differential equations that govern the total deflection y (x, t) and the bending slope ψ (x, t) of a beam vibrating freely are

$$\left.\begin{aligned} kGA\left(\frac{\delta^2 y}{\delta x^2}-\frac{\delta \psi}{\delta x}\right) &= \frac{A\gamma}{g}\cdot\frac{\delta y^2}{\delta t^2} \\ EI\,\frac{\delta^2 \psi}{\delta x^2} + kGA\left(\frac{\delta y}{\delta x}-\psi\right) &= \frac{I\gamma}{g}\cdot\frac{\delta^2 \psi}{\delta t^2} \end{aligned}\right\} \qquad (1)$$

where γ = weight per unit volume
A = cross-sectional area
E = modulus of elasticity
G = shear modulus
g = acceleration due to gravity
k = shear co-efficient
I = moment of inertia

* Professor and Head, Department of Civil Engineering, Indian Institute of Technology, Kanpur.
** Senior Scientific Officer, Structural Engineering Research Centre, Madras.

Here, shear slope, $\phi (x, t) = \frac{\delta y}{\delta x} - \psi$

bending moment, $M (x, t) = -EI \frac{\delta \psi}{\delta x}$

shear force, $Q (x, t) = kGA \left(\frac{\delta y}{\delta x} - \psi\right)$

For the simplest configurations, the boundary conditions are the following :

Hinged end : $y = 0 ; \frac{\delta \psi}{\delta x} = 0$

Clamped end : $y = 0 ; \psi = 0$

Free end : $\frac{\delta \psi}{\delta x} = 0 ; \frac{\delta \psi}{\delta x} - \psi = 0$

Eliminating ψ and y from equations (1), the following two uncoupled equations in y and ψ are obtained

$$\left.\begin{aligned} EI \frac{\delta^4 y}{\delta x^4} + \frac{\gamma A}{g} \cdot \frac{\delta^2 y}{\delta t^2} - \left(\frac{\gamma I}{g} + \frac{EI}{gk} \cdot \frac{\gamma}{g}\right) \frac{\delta^4 y}{\delta x^2 \delta t^2} + \frac{I\gamma}{g} \cdot \frac{\gamma}{gkG} \cdot \frac{\delta^4 y}{\delta t^4} = 0 \\ EI \frac{\delta^4 \psi}{\delta x^4} + \frac{\gamma A}{g} \cdot \frac{\delta^2 \psi}{\delta t^2} - \left(\frac{\gamma I}{g} + \frac{EI}{gk} \cdot \frac{\gamma}{G}\right) \frac{\delta^4 \psi}{\delta x^2 \delta t^2} + \frac{\gamma I}{g} \cdot \frac{\gamma}{gkG} \cdot \frac{\delta^4 \psi}{\delta t^4} = 0 \end{aligned}\right\} \quad (2)$$

Let us take the solutions of equations (1) and (2) in the form

$$\left.\begin{aligned} y (x, t) &= \gamma (x) e^{ipt} \\ \psi (x, t) &= \psi (x) e^{ipt} \end{aligned}\right\} \quad (3)$$

Substituting equations (3) into (1) and (2) and eliminating time t, we obtain,

$$\left.\begin{aligned} S^2 \psi'' - (1 - b^2 r^2 s^2) \psi + \frac{Y'}{L} &= 0 \\ Y'' + b^2 s^2 Y - L\psi' &= 0 \\ \gamma^{iv} + b^2 (r^2 + s^2) Y'' - b^2 (1 - b^2 r^2 s^2) Y &= 0 \\ \psi^{iv} + b^2 (r^2 + s^2) \psi'' - b^2 (1 - b^2 r^2 s^2) \psi &= 0 \end{aligned}\right\} \quad (4)$$

where $b^2 = \frac{1}{EI} \cdot \frac{\gamma A}{g} L^4 p^4$

$r^2 = \frac{I}{AL^2}$

$s^2 = \frac{EI}{kGAL^2}$

In the above equations prime indicates derivative with respect to $\xi = x/L$. The dimensionless parameter b is related to natural frequencies of the beam and r and s are measures of the effects of rotary inertia and shear deformations respectively.

There are two sets of solutions of equations (4) and (6). They are,

Case (1) : when $[(r^2 - s^2) + 4/b^2]^{1/2} > (r^2 + s^2)$

$$\left.\begin{aligned} Y &= C_1 \cosh b \alpha \xi + C_2 \sinh b \alpha \xi + C_3 \cos b \beta \xi + C_4 \sin b \beta \xi \\ \psi &= C'_1 \sinh b \alpha \xi + C'_2 \cosh b \alpha \xi + C'_3 \sin b \beta \xi + C'_4 \cos b \beta \xi \end{aligned}\right\} \quad (5)$$

where
$$\alpha = \frac{1}{\sqrt{2}}\left[\{ (r^2 - s^2)^2 + 4/b^2 \}^{1/2} - (r^2+s^2)\right]^{1/2}$$
$$\beta = \frac{1}{\sqrt{2}}\left[\{ (r^2 - s^2)^2 + 4/b^2 \}^{1/2} + (r^2+s^2)\right]^{1/2}$$

Case (2) - when $[(r^2 - s^2)^2 + 4/b^2]^{1/2} < (r^2+s^2)$

$$\left.\begin{aligned} Y &= C_1 \cos b\,\alpha'\,\xi + i\,C_2 \sin b\,\alpha'\,\xi + C_3 \cos b\,\beta\,\xi + C_4 \sin b\,\beta\,\xi \\ \psi &= i\,C'_1 \sin b\,\alpha'\,\xi + C'_2 \sin b\,\alpha'\,\xi + C'_3 \sin b\,\beta\,\xi + C'_4 \cos b\,\beta\,\xi \end{aligned}\right\} \quad (6)$$

where $\alpha' = -\,i\,\alpha$

Only one-half of the constants in equations (5) and (6) are independent. They are related by equations (4) as follows :

$$\left.\begin{aligned} C_1' &= \frac{b}{L}\cdot\frac{\alpha^2+s^2}{\alpha}\cdot C_1 \\ C_2' &= \frac{b}{L}\cdot\frac{\alpha^2+s^2}{\alpha}\cdot C_2 \\ C_3' &= -\frac{b}{L}\cdot\frac{\beta^2-s^2}{\beta}\cdot C_3 \\ C_4' &= \frac{b}{L}\cdot\frac{\beta^2-s^2}{\beta}\cdot C_4 \end{aligned}\right\} \quad (7)$$

FREQUENCY EQUATIONS

The application of appropriate boundary conditions and continuity conditions at the centre, equations (5) or (6) will give four homogeneous algebraic equations for four constants C_1 to C_4. The non-trivial solution will give transcendental equation to calculate the natural frequencies.

Since the boundary conditions and the location of the mass are symmetric, and if we consider the symmetric and anti-symmetric modes separately, we can consider half of the beam as follows :

Case (1) : Simply supported beam with central mass.

(a) Symmetric Modes :

The boundary and continuity conditions for the beam are :

$$\text{at} \quad x = 0 \qquad y = 0, \qquad \frac{\delta\psi}{\delta x} = 0$$
$$\text{at} \quad x = L/2 \qquad \psi = 0$$
$$Q = kGA\left(\frac{\delta y}{\delta x} - y\right) = -\frac{M}{2}\cdot\frac{\delta^2 y}{\delta t^2} \quad (8)$$

where M = mass of the attached mass

Satisfying these conditions, will lead to frequency equation
$$(1+\zeta) = \frac{M}{m}\cdot\frac{b}{2}\left(\beta \tan\frac{b\beta}{2} - \alpha\,\zeta \tanh\frac{b\alpha}{2}\right) \quad (9)$$

for $[(r^2 - s^2)^2 + 4/b^2]^{1/2} > (r^2+s^2)$

$$(1+\zeta) = \frac{M}{m} \cdot \frac{b}{2}\left(\beta \tan \frac{b\beta}{2} + \alpha' \zeta \tan \frac{b\alpha'}{2}\right) \quad (10)$$

for $[(r^2 - s^2)^2 + 4/b^2]^{1/2} < (r^2 + s^2)$

where $\zeta = \frac{\beta^2 - r^2}{\alpha^2 + s^2} = \frac{\alpha^2 + r^2}{\alpha^2 + s^2}$

(b) Asymmetric Modes :

The boundary and continuity conditions for the beam

at $x = 0, \quad y = 0$

$$\frac{\delta\psi}{\delta x} = 0$$

at $x = L/2, \quad y = 0$

$$M_o = - EI \frac{\delta\psi}{\delta x} = \frac{B}{2} \frac{\delta^2\psi}{\delta t^2} \quad (11)$$

where B = moment of inertia of the attached mass

Frequeney equation in this case will be,

$$\left.\begin{aligned} (1+\zeta) &= \frac{4B}{mL^2} \cdot \frac{b}{8}\left(\frac{1}{\alpha} \coth \frac{b\alpha}{2} - \frac{\zeta}{\beta} \cot \frac{b\beta}{2}\right) \\ &\text{for } [(r^2 - s^2)^2 + 4/b^2]^{1/2} > (r^2 + s^2) \\ (1+\zeta) &= \frac{4B}{mL^2}\, b/8 \left(- 1/\alpha' \cot \frac{b\alpha'}{2} - \zeta/\beta \cot \frac{b\beta}{2}\right) \\ &\text{for } [(r^2 - s^2)^2 + 4/b^2]^{1/2} < (r^2 + s^2) \end{aligned}\right\} \quad (12)$$

Case (2) : Clamped beam with central mass

(a) Symmetric Modes

The boundary and continuity conditions are

at $x = 0 \quad y = 0 \quad , \quad \psi = 0$

at $x = L/2 \quad \psi = 0$

$$Q = k\ G A \left(\frac{\delta y}{\delta x} - \psi\right) = - \frac{M}{2} \frac{\delta^2 y}{\partial t} \quad (13)$$

These conditions will lead to the frequency equation :

$$\frac{1}{\beta} (1 + \zeta) \left[\sin h \frac{b\alpha}{2} \cos \frac{b\beta}{2} + \lambda \zeta \cosh \frac{b\alpha}{2} \sin \frac{b\beta}{2}\right]$$

$$= \frac{M}{m} \cdot \frac{b}{2} \left[2 \lambda \zeta \left(1 - \cosh \frac{b\alpha}{2} \cos \frac{b\beta}{2}\right) + \left(1 - \lambda^2\zeta^2 \right) \sin \frac{b\beta}{2} \sin h \frac{b\alpha}{2}\right] \quad (14)$$

for $[(r^2 - s^2)^2 + 4/b^2]^{1/2} > (r^2 + s^2)$

$$\frac{1}{\beta}(1+\zeta)\left[\sin\frac{b\alpha'}{2}\cos\frac{b\beta}{2}+\lambda'\zeta\cos\frac{b\alpha'}{2}\sin\frac{b\beta'}{2}\right]$$

$$=\frac{M}{m}\cdot\frac{b}{2}\left[2\lambda'\zeta\,(1-\cos b\alpha'/2\cos b\beta/2)+(1+\lambda'^2\zeta^2)\sin\frac{b\alpha'}{2}\sin\frac{b\beta}{2}\right] \qquad (15)$$

for $[(r^2-s^2)^2+4/b^2]^{1/2} < (r^2+s^2)$

(b) Asymmetric modes

In this case the boundary and continuity conditions are :

at $x = 0$ $\quad y = 0$ and $\psi = 0$

at $x = L/2$ $\quad y = 0$

$$M_0 = -\,EI\frac{\delta\psi}{\delta x} = \frac{B}{2}\frac{\delta^2\psi}{\delta t^2}$$

The corresponding frequency equations are :

$$(\alpha+\lambda\zeta\beta)\left(\cos\frac{b\beta}{2}\sinh\frac{b\alpha}{2}-\frac{1}{\lambda\zeta}\sin\frac{b\beta}{2}\cosh\frac{b\alpha}{2}\right)$$

$$=\frac{4B}{mL^2}\frac{b}{8}\left[2\cosh\frac{b\alpha}{2}\cos\frac{b\beta}{2}-2+(\lambda\zeta-1/\lambda\zeta)\sinh\frac{b\alpha}{2}\sin\frac{b\beta}{2}\right] \qquad (16)$$

for $[(r^2-s^2)^2+4/b^2]^{1/2} > (r^2+s^2)$

$$(\alpha'+\lambda'\zeta\beta)\left[\sin\frac{b\alpha'}{2}\cos\frac{b\beta'}{2}+\frac{1}{\lambda'\zeta}\cdot\cos\frac{b\alpha'}{2}\sin\frac{b\beta}{2}\right]$$

$$=\frac{M}{mL^2}\frac{b}{8}\left[2-2\cos\frac{b\alpha'}{2}\cos\frac{b\beta}{2}+\left(\lambda'\zeta+\frac{1}{\lambda'\zeta}\right)\sin\frac{b\alpha'}{2}\sin\frac{b\beta}{2}\right] \qquad (17)$$

when $[(r^2-s^2)^2+4/b^2]^{1/2} < (r^2+s^2)$

NUMERICAL SOLUTION OF THE FREQUENCY EQUATIONS :

For a given beam with r and s known the b_i ($i = 1, 2, 3......$) can be formed from the appropriate frequency equations and the corresponding p_i are then calculated from the equation :

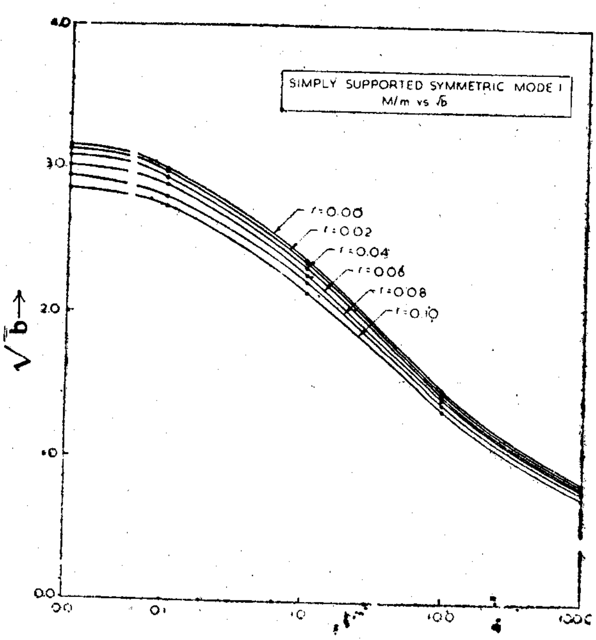
$$p_i^2 = b_i^2\,\frac{EIg}{AL^4}$$

The frequency equations are highly transcendental. In order to solve them a frequency chart is obtained from the solution of these trancendental frequency equations for various combinations of r and s and various boundary conditions. It is assumed that $k = 2/3$ and $E/G = 8/3$. Hence $E/kG = 4$ and $s = 2r$. r is assumed to vary from 0 to 0·10 and M/m and $4B/mL^2$ are taken from 0.00 to 100 and 0.00 to 10.00 respectively.

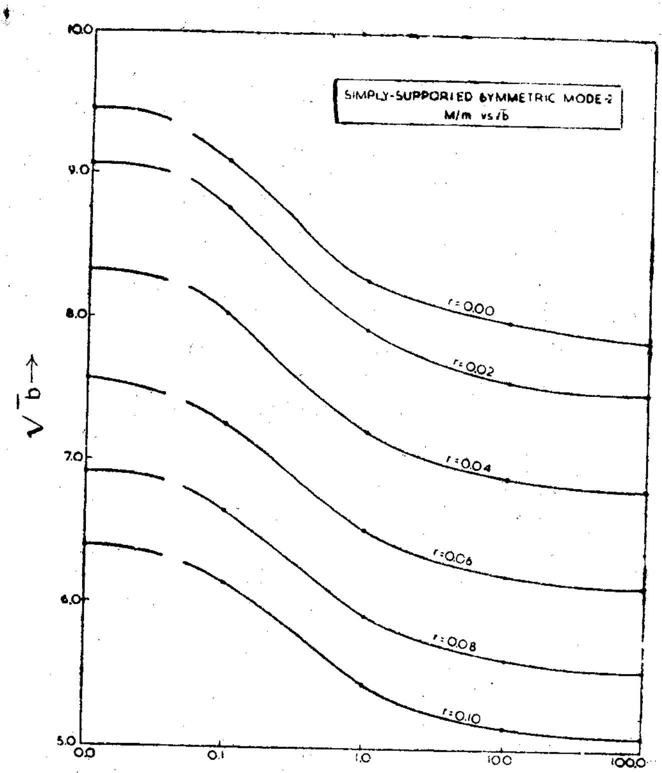
The frequency equations are solved on IBM 1620 computer. The solutions for the first two symmetric and antisymmetric modes are given. The results are given in figures 1to8.

REFERENCES

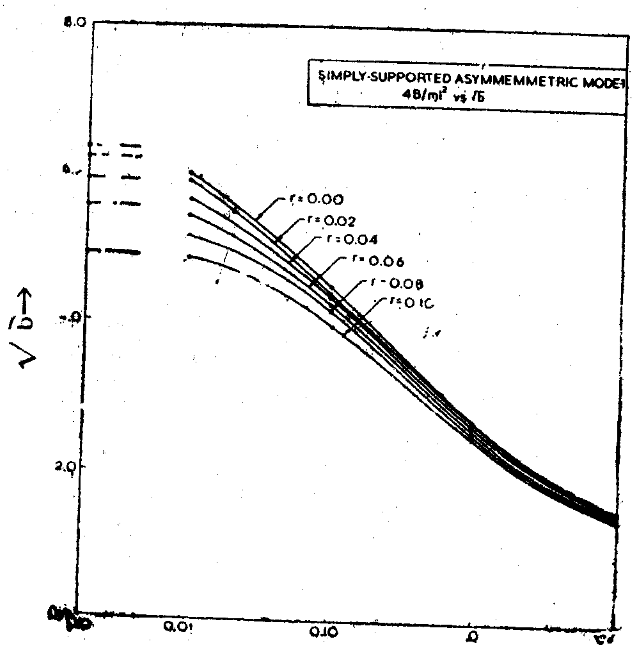
1. Baker, W. E. "Vibration Frequencies for Uniform Beams with Central Masses", Journal of Applied Mechanics Trans. ASME., Vol. 31, June, 1964.
2. Chen, Yu, "On the Vibration of Beams or Rods Carrying a Concentrated Mass", Journal of Applied Mechanics Trans. ASME, Vol 85, 1963
3. Maltbaek, J.C., "The Influence of a Concentrated Mass on the Free Vibrations of a Uniform Beam". Int. Journal, Mech. Sci., Pergamon Press Ltd., Vol. 3, 1961. pp. 197–218.



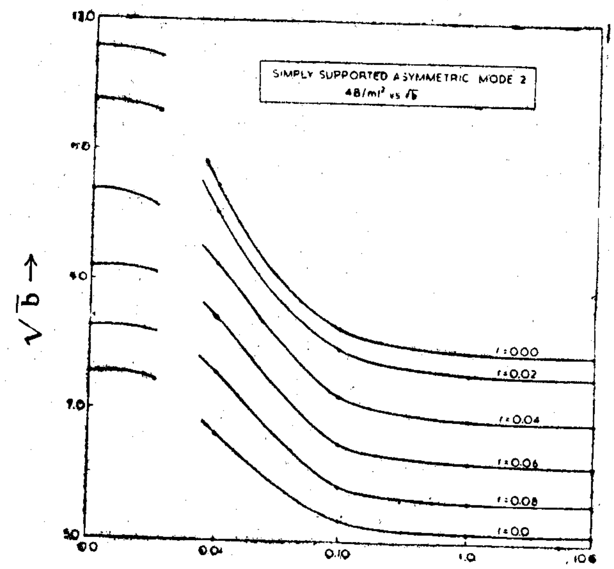
M/m→
Figure 1



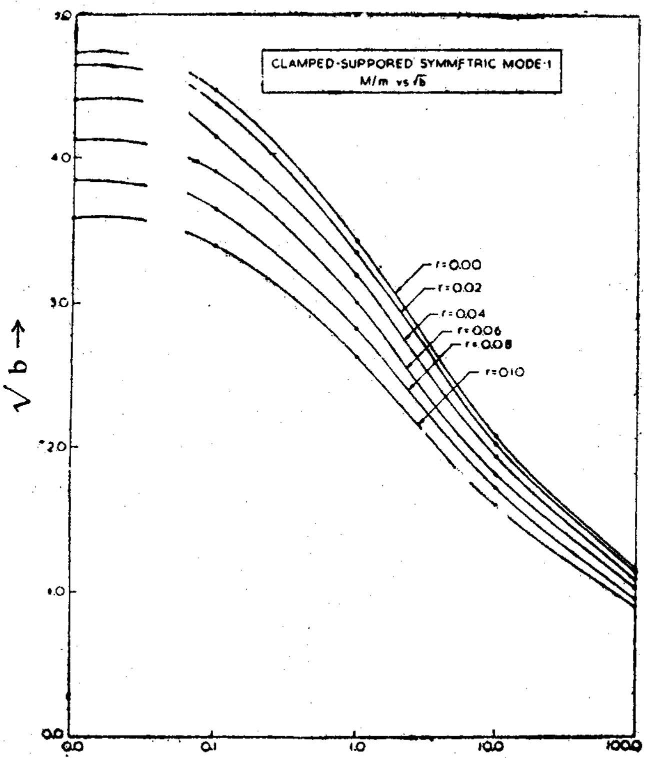
M/m→
Figure 2



$\frac{4\,B}{ml^2}\rightarrow$
Figure 3

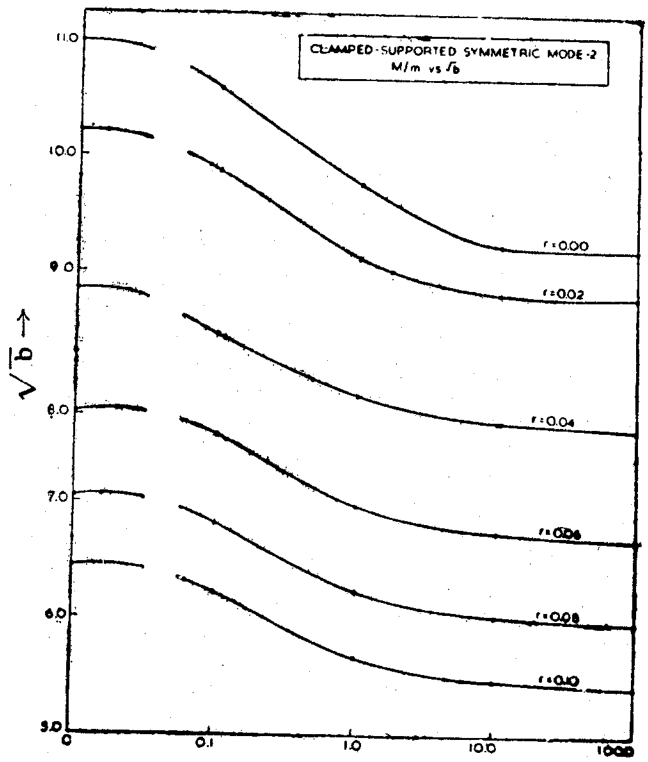


$\frac{4\,B}{ml^2}\rightarrow$
Figure 4



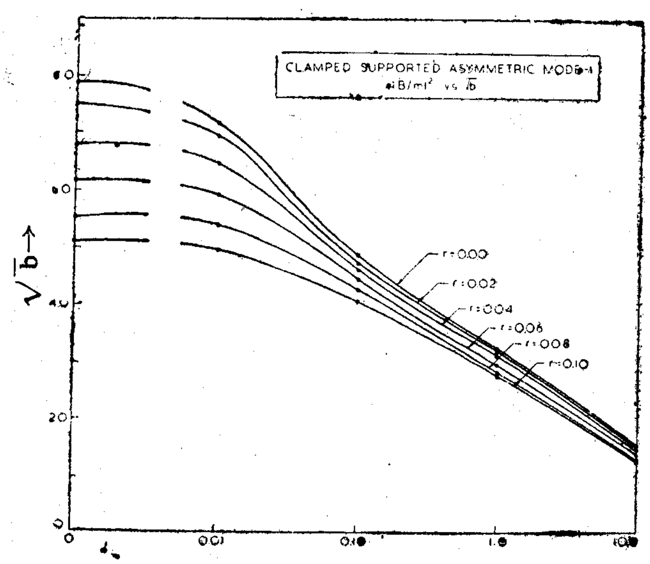
M/m→

Figure 5



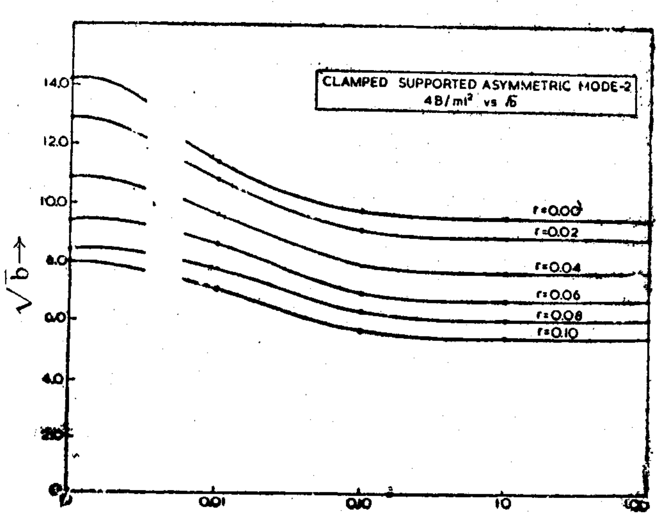
M/m→

Figure 6



$\frac{4B}{ml^2} \rightarrow$

Figure 7



$\frac{4B}{ml^2} \rightarrow$

Figure 8

Paper No. 91. Bulletin of the Indian Society of Earthquake Technology, Vol.VII, No. 2, June 1970, pp 97–101

PSEUDO EARTHQUAKES DERIVED FROM RECORDED ACCELEROGRAMS

Brijesh Chandra* Member and A. R. Chandrasekaran** Member

INTRODUCTION

Accelerogram forms an important part of data in computing structural response due to an earthquake. For design of structures in seismic zones, the accelerogram of a probable future shock needs to be predicted This is a very difficult task as instrumental data for a particular site would normally not be available. A designer would therefore like to make an estimate of accelerogram characteristics based on experience in similar conditions elsewhere. Methods of generating artificial earthquakes have been developed [3,4] for use in various situations. Pseudo earthquakes can also be derived from recorded earthquakes. One such attempt[1] was made by shuffling acceleration ordinates of the accelerogram. This changed the time of occurence of the impulses while the frequency contents of the accelerogram were not much altered. Response spectra obtained for such shuffled accelerogram indicated that the spectral quantities are not much affected by this process. Such modifications, therefore, do not seem to provide a variety of data to be used in different situations.

It has been observed[2] that as the distance from epicentre increases, the accelerogram shows lower frequency components as also the acceleration amplitudes get attenuated. If an accelerogram has been obtained at a certain distance from the epicentre, it would be reasonable to assume that at greater distances than this, the frequencies present shall be lower and vice versa.

In connection with predicting an accelerogram for a dam site it was desired that it has the important characteristics of two of the strongest recorded earthquakes viz. El Centro May 18, 1940 and Koyna Dec. 11, 1967. The method of changing the time base was proposed by the authors so that the response spectra of the pseudo earthquake had the desired features in the frequency range of interest. Tsai[6], working independently, has also proposed such a method of changing time base to derive pseudo earthquakes.

A definition of intensity of the earthquakes corresponds to the area under the curve of spectral velocity versus period. A different interpretation of this definition has been given which may be useful for multiple degree and continuous systems.

INFLUENCE OF CHANGE IN TIME BASE

The equation of motion of an idealized single degree oscillator with mass m, damping c and stiffness k is given by.

$$m\ddot{x}\ c\dot{x} + + kx = -m\ddot{y}(t) \tag{1}$$

in which x is the relative displacement of mass with respect to ground and y is the ground

On page 97, equation No. : (1) should be as follows

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{y}(t) \tag{1}$$

Modification of time base t by a factor λ can be accounted for by putting $\tau = \lambda t$, in which τ is a new time parameter. Eqn. (1) would then be as follows :

$$m \lambda^2 \frac{d^2x}{d\tau^2} + c \lambda \frac{dx}{d\tau} + kx = -m \lambda^2 \frac{d^2y}{d\tau^2} \quad (2)$$

Dividing throughout by λ^2 one obtains,

$$m \frac{d^2x}{d\tau^2} + \frac{c}{\lambda} \frac{dx}{d\tau} + \frac{k}{\lambda^2} x = -m \frac{d^2y}{d\tau^2} \quad (3)$$

Solution of Eqn. (3) in terms of spectral quantities of modified time-based accelerogram and those of original one can be written as follows :

$$(S_d^*)_{\lambda T} = (S_d)_T . \lambda^2 \quad (4a)$$

$$(S_v^*)_{\lambda T} = (S_v)_T . \lambda \quad (4b)$$

$$(S_a^*)_{\lambda T} = (S_a)_T . \quad (4c)$$

in which asterik indicates modified spectral quantities.

STUDY OF RESPONSE SPECTRA

Figs 1—3 show the spectral quantities S_a, S_v and S_d respectively for 5% damping for two recorded accelerograms Koyna Dec. 11. 1967 Longitudinal component (KL) and EL Centro May 18, 1940 N-S component (EC). Response parameters for KL with $\lambda = 1.5$ and EC with $\lambda = 0.66$ have also been plotted in the respective figures. It is interesting to note that spectral quantities for KL accelerogram compare very well with those of EC with $\lambda = 0.66$. Also, spectral quantities for EC accelerogram compare well with those of KL with $\lambda = 1.5$. This is a clear indication that accelerogram recorded at one site could be used for determining spectral quantities at another place by a suitable choice of factor λ. This factor could be obtained as the ratio of spectral velocities of two earthquakes in the range in which S_v tends to be flat. This is clear from eqn. (4 b).

It is thus seen that elongation or contraction of the time base of accelerogram results in change of the intensity potential of an earthquake. In this context, the effect of λ on 'intensity' has also been studied.

INTENSITY OF SHOCK

Intensity of earthquake has been used by various authors as a means to decribe the potential of a shock, Among these, peak ground acceleration, peak ground displacement, spectral intensity, root mean square acceleration, and time averaged r. m. s. value of ground acceleration have been suggested[5]. From the point of view of structural designer, the spectral intensity given by Housner seems to be an objective method of defining earthquake intensity. But all these definitions have limited use for single degree freedom systems and may not be applicable for systems with multiple degrees or for continuous systems.

A designer would be interested in knowing about spectral quatities in fundamental mode and a few higher modes. In such cases the quantity of interest would be the area of Sv V_s T curve upto the fundamental period. With this in view, the following parameter called 'Velocity Intensity' of shock is defined as,

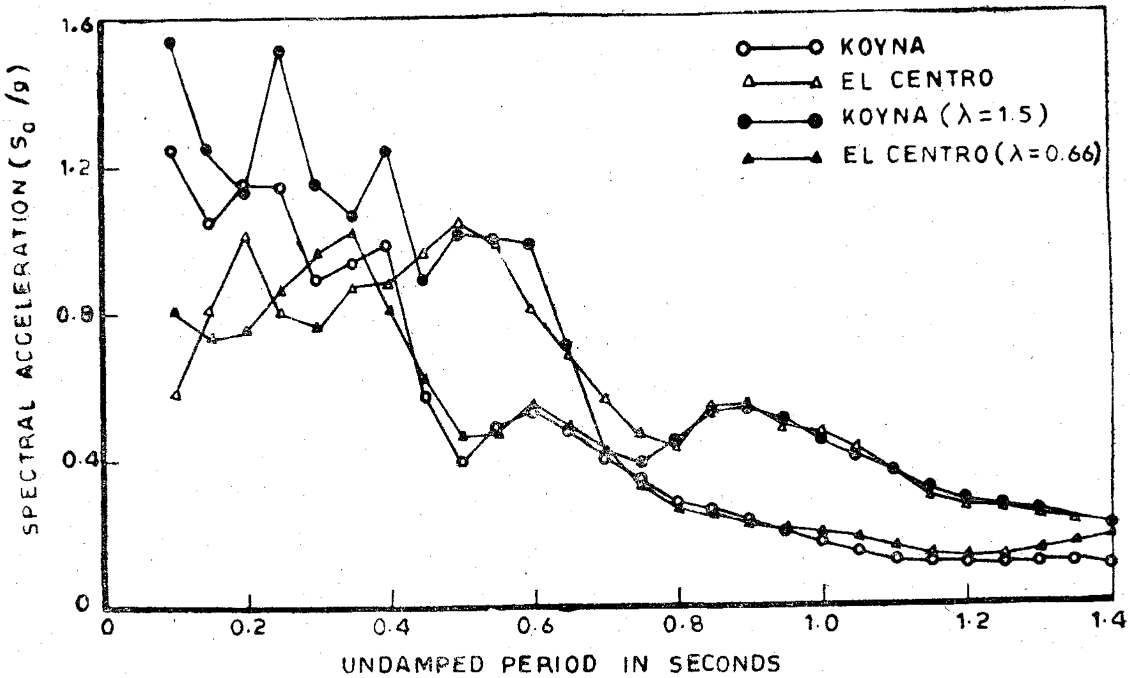


Figure 1. Acceleration Spectra for Koyna, El Centro, Modified Koyna and Modified El Centro Shocks (Damping = 5%)

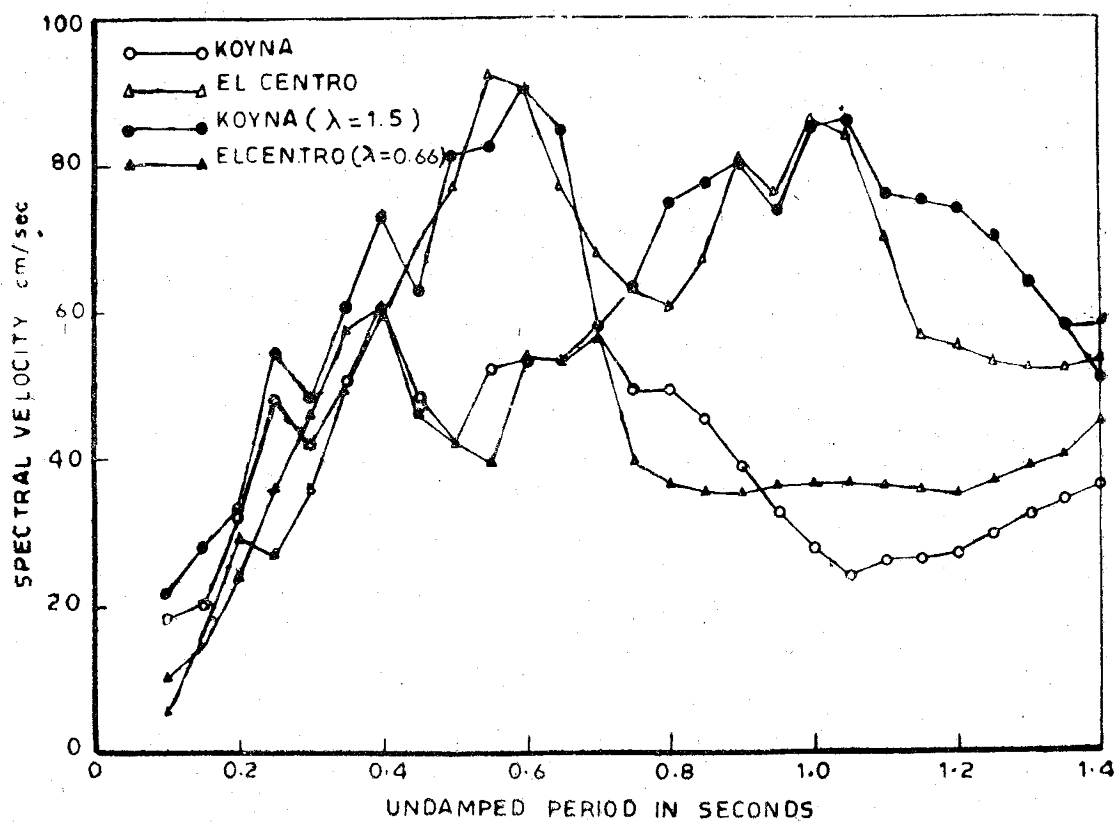


Figure 2. Velocity Spectra for Koyna, El Centro, Modified Koyna and Modified El Centro Shocks (Damping = 5%)

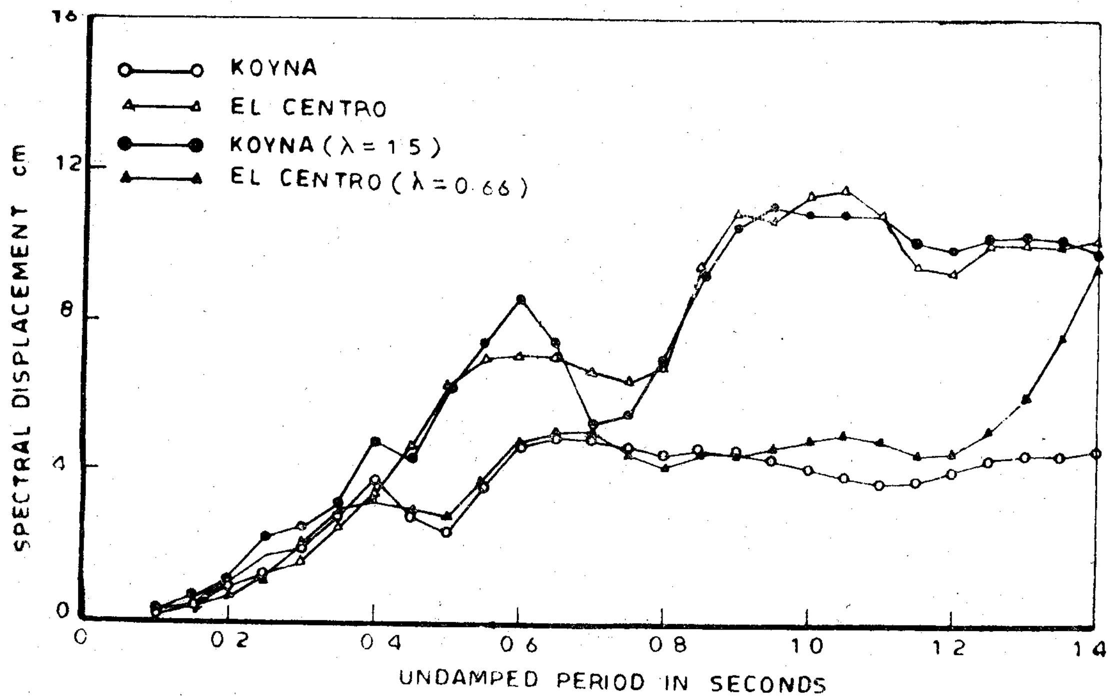


Figure 3. Displacement Spectra for Koyna, El Centro, Modified Koyna and Modified El Centro Shocks (Damping = 5%)

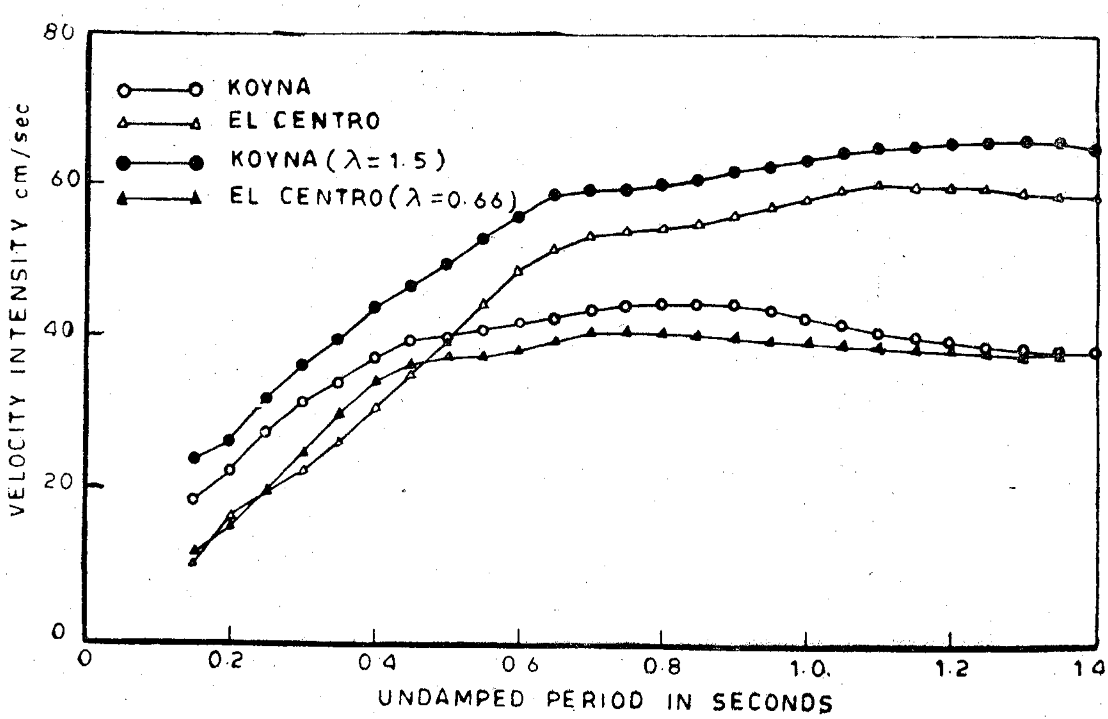


Figure 4. Velocity Intensity for Koyna, El Centro, Modified Koyna and Modified El Centro Shocks (Damping = 5%)

$$VI = \frac{1}{T-0.1} \int_{0.1}^{T} S_v (T. \zeta)\, dT \tag{5}$$

Fig. 4 shows a plot of VI versus T for accelerograms KL, EC, KL with $\lambda = 1.5$ and EC with $\lambda = 0.66$ and corresponding to 5% damping. Again, it will be noticed that VI plots for KL and EC with $\lambda = 0.66$ as also for EC and KL with $\lambda = 1.5$ are in reasonably good agreement.

For some of the examples of multiple degree systems worked out by the authors, the ratio of response as obtained for two earthquakes corresponds to the ratio of VI for those two earthquakes at the appropriate fundamental period.

CONCLUSIONS

Elongation or contraction of time base of accelerograms as described in this paper may be used for generation of pseudo earthquakes.

It is felt that velocity intensity of the earthquake as defined here will be found very helpful in comparing response of multiple degree and continuous systems under different earthquakes.

ACKNOWLEDGEMENTS

This paper is being published with the permission of Director, S.R.T.E.E. Roorkee.

REFERENCES

1. Chandrasekaran, A.R. and Saini, S.S , "Shuffling of Earthquake Data", Bu letin, Indian Society of Earthquake Technology, Vol. IV, No, 3, Sept. 1967.
2. Chandra. B. "A Method of Estimating Response of Structures During Earthquakes", Bulletin, Indian Society of Earthquake Technology, Vol. VI No. 3, Sept. 1969.
3. Housner, G.W. and Jennings, P.C., "Generation of Artificial Earthquakes", Proc. ASCE. Vol. 90, EM 1, 1964.
4. Jennings, P.C., Housner, G. W. and Tsai, N.C., "Simulated Earthquake Motion", Earthquake Engineering Research Laboratory (Report), C.I.T , Pasadena, April 1968.
5. Lin. Shih-Chi, "On Intensity Definitions of Earthquakes" Technical Note, Proc. ASCE, Vol. 95, ST5, May 1969.
6. Tsai, N.C., "Transformation of time Axes of Accelegrograms", Technical Note, Proc. ASCE, Vol. 95, EM 3, June 1969.