STUDY OF SHEAR BEAMS UNDER DYNAMIC LOADS

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SYNOPSIS

Beams in which deformations due to a load are essentially due to shearing action are known as 'Shear Beams'. In practice, cantilever beams of very large cross sectional area and of small length have predominantly shear deformation. Such type of structures, however, would have very high natural freqnency and therefore would be very little affected by ground motion.

The equation of motion of a shear structure is of second degree and so is the equation of motion of multistoreyed framed structure. Therefore, there is a possibility that cantilever shear beams could be theoretical models of multistoreyed structures. In this paper, it is proposed to study shear beams and compare them with multistoreyed frames.

1. Basic Equations of the Problem

The following assumptions are made in solving the problem: The material of which the beam is made is homogeneous, isotropic and behaves elastically. The deformations are in shear only.

Considering the equilibrium of elastic and inertia forces, (adopting the procedure outlined by Rogers (1959)).

$$V = -\sigma' AG\frac{\partial y}{\partial x} \qquad 1.1\dagger$$

$$\frac{\partial v}{\partial x} = -\rho A\,\frac{\partial^2 y}{\partial t^2} + w\,(x, t) \qquad 1.2$$

From 1.1 and 1.2

$$-\frac{\partial}{\partial x}\left(\sigma' AG\frac{\partial y}{\partial x}\right) = -\rho A\,\frac{\partial^2 y}{\partial t^2} + w(x, t) \qquad 1.3$$

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† The letter symbols adopted for use in this paper are defined and are listed alphabatically in the Appendix.

FREE VIBRATION

For the free vibration problem, w(x,t) will be equal to zero. If a harmonic solution in time, with circular frequency p, is assumed, then equation 1.3 reduces to

$$\frac{d}{dx}\left(\sigma' AG\frac{dy}{dx}\right)+\rho Ap^2y=0 \qquad 1.4$$

The solution of equation 1.4 would give the frequencies p_r and corresponding mode shapes $\phi_r(x)$. Since a beam has infinite degrees of freedom, there will be infinite frequencies.

In general, the free vibration solution has the form

$$y(x,t)=\sum_{r=1}^{\infty} \phi_r(x) \,.\, D_r \,.\, \sin(p_r t+\phi_r) \qquad 1.5$$

GROUND MOTION EXCITATION

For ground motion excitation,

$$w(x,t)=\rho(x) \,.\, A(x) \,.\, a(t) \qquad 1.6$$

Substituting 1.6 in 1.3 and if Z represents relative displacement, with respect to the base, at any section x, then

$$-\frac{\partial}{\partial x}\left(\sigma' AG\frac{\partial Z}{\partial x}\right)=-\rho(x) \,.\, A(x) \,.\, \frac{\partial^2 Z}{\partial t^2}+\rho(x) \,.\, A(x) \,.\, a(t) \qquad 1.7$$

Let $Z(x,t)=\sum_{r=1}^{\infty} \phi_r(x)\, \xi(t)$ 1.8

Substituting 1.8 in 1.7

$$-\sum_{r=1}^{\infty} \xi_r(t)\left\{\frac{d}{dx}\left(\sigma' AG\frac{d\phi_r}{dx}\right)\right\}=-\sum_{r=1}^{\infty} m(x) \,.\, \ddot{\xi}_r(t) \,.\, \phi_r(x)+m(x) \,.\, a(t) \qquad 1.9$$

where m(x) has been taken equal to $\rho(x) \,.\, A(x)$.

From 1.4 and 1.8,

$$\sum_{r=1}^{\infty}\left\{\frac{d}{dx}\left(\sigma' AG\frac{d\phi_r}{dx}\right)\right\} \xi_r(t)=-\sum_{r=1}^{\infty} m(x) \,.\, p^2_r \,.\, \phi_r(x) \,.\, \xi_r(t) \qquad 1.10$$

combining 1.9 and 1.10,

$$\sum_{r=1}^{\infty} (\ddot{\xi}_r+p_r^2\xi_r) \,.\, m(x) \,.\, \phi_r(x)=m(x) \,.\, a(t) \qquad 1.11$$

Expressing the right hand side of 1.11, in terms of mode functions, $\phi_r(x)$, and making use of the relationship of orthogonality of modes, namely

$$\int_0^H m(x) \cdot \phi_r(x) \cdot \phi_s(x) \cdot dx = 0 \quad \text{for } r \neq s \text{ and}$$

$$m(x) \cdot a(t) = \sum_{r=1}^{\infty} a(t) \cdot m(x) \cdot \phi_r(x) \cdot \frac{\int_0^H \phi_r(x) \cdot m(x)\, dx}{\int_0^H (\phi_r(x))^2 \cdot m(x)\, dx} \tag{1.12}$$

from 1.11 and 1.12

$$\ddot{\xi}_r + p_r^2 \xi_r = a(t) \cdot \frac{\int_0^H \phi_r(x) \cdot m(x)\, dx}{\int_0^H (\phi_r(x))^2 \cdot m(x)\, dx} \tag{1.13}$$

The solution of 1.13 is

$$\xi_r = -\frac{1}{p_r} \int_0^t a(t) \cdot \frac{\int_0^H \phi_r(x) \cdot m(x)\, dx}{\int_0^H (\phi_r(x))^2 \cdot m(x)\, dx} \cdot \sin p_r(t-\tau)\, d\tau \tag{1.14}$$

from 1.8 and 1.14

$$Z(x, t) = -\sum_{r=1}^{\infty} \frac{\phi_r(x)}{p_r} \frac{\int_0^H \phi_r(x) \cdot m(x)\, dx}{\int_0^H (\phi_r(x))^2 \cdot m(x)\, dx} \int_0^t a(t) \cdot \sin p_r(t-\tau)\, d\tau \tag{1.15}$$

If there is damping in the system such that mode superposition is still applicable, then

$$Z(x, t) = -\sum_{r=1}^{\infty} \frac{\phi_r(x)}{p_{dr}} \cdot \frac{\int_0^H \phi_r(x) \cdot m(x)\, dx}{\int_0^H (\phi_r(x))^2 \cdot m(x) dx} \int_0^t a(t) \cdot e^{-\zeta_r p_r (t-\tau)} \sin p_{dr}(t-\tau)\, d\tau \tag{1.16}$$

From 1.16, the maximum relative displacement at any section x, due to r^{th} mode of vibration could be expressed as

$$|Z_x^{(r)}| = -B_x^{(r)} \cdot \frac{1}{p_r} \cdot (S_v)_r \tag{1.17}$$

From 1.1, the shear force, V, at any section x is,

$$V_x^{(r)} = \sigma' AG \frac{dZ_x^{(r)}}{dx} \tag{1.18}$$

The bending moment, MT, at any section x, is

$$MT_x^{(r)} = \int_0^h V_x^{(r)} \cdot dx \tag{1.19}$$

where h is measured from the free end.

2. Theoretical Solution

To find the effect of ground motion on a shear beam, its frequencies and mode shapes are to be determined. In certain cases, where the properties of the beam vary in a regular fashion along the beam, theoretical solutions are possible (Conway, 1948). Some of these cases are discussed below. Where theoretical solutions are cumbersome or impossible, numerical methods could be adopted with success.

Let the area of cross section 'A' be a function of the length of the beam, that is

$$A=f(x) \qquad 2.1$$

Substituting 2.1 in 1.4

$$\frac{d^2y}{dx^2}+\frac{f'(x)}{f(x)}\frac{dy}{dx}+\gamma^2y=0 \qquad 2.2$$

where prime denotes differentiation and

$$\gamma^2=\frac{\rho p^2}{\sigma' G} \qquad 2.2a$$

UNIFORM BEAM

Here f(x) is constant. Hence 2.2 reduces to

$$\frac{d^2y}{dx^2}+\gamma^2y=0 \qquad 2.3$$

For a cantilever beam with boundary conditions,

$$\left.\begin{aligned} y&=0 \text{ at } x=H \\ \text{and} \quad \frac{\partial y}{\partial x}&=0 \text{ at } x=0 \end{aligned}\right\} \quad \text{for all } t \qquad 2.4$$

the frequency equation is given by

$$\cos \gamma H=0=\cos \frac{2r-1}{2}\pi \qquad 2.5$$

that is $\gamma H=\frac{2r-1}{2}\pi$ 2.6

where $r=1,2,3\ldots\infty$

from 2.2a,

$$p_r=\frac{2r-1}{2}\cdot\frac{\pi}{H}\cdot\sqrt{\frac{\sigma' G}{\rho}} \qquad 2.7$$

The mode shape is given by

$$\phi_r(x)=\cos\frac{(2r-1)\cdot\pi\cdot x}{2H} \qquad 2.8$$

The general free vibration solution is given by

$$y(x,t)=\sum_{r=1}^{\infty}\left\{D_r \,.\, \sin(p_r t+\theta_r)\right\} \,.\, \cos\frac{(2r-1)\,.\,\pi\,.\,x}{2H} \qquad 2.9$$

NON-UNIFORM BEAM

Consider the case when

$$A=f(x)=A_0\left(a+\frac{bx}{H}\right)^s \qquad 2.10$$

where x is measured from free end and A_0, a, b and s are constants.

Then 1.4 could be written as

$$\frac{d^2y}{dx^2}+\frac{s\,.\,b/H}{a+b(\frac{x}{H})}\,.\,\frac{dy}{dx}+\gamma^2 y=0 \qquad 2.11$$

$$\text{Let}\quad \gamma H\left(a+\frac{b}{H}x\right)=\psi \qquad 2.12$$

then from 2.11 and 2.12,

$$\frac{d^2y}{d\psi^2}+\frac{s}{\psi}\frac{dy}{d\psi}+y=0 \qquad 2.13$$

This is Bessal equation of order q where

$$q=\frac{1-s}{2}$$

The solution of 2.13 has the form

$$y(x)=D_1\,.\,J_q(x)+D_2\,.\,Y_q(x) \qquad 2.14$$

In particular, consider the case when s=2. This corresponds to the case of linearly tapering beam, then

$$q=\frac{1-s}{2}=-\tfrac{1}{2}$$

$$\text{and}\quad y(\psi)=D_1\,.\,J_{-\frac{1}{2}}(\psi)+D_2\,.\,J_{\frac{1}{2}}(\psi) \qquad 2.15$$

$$=\frac{1}{\psi}(D'_1\cos\psi+D_2'\sin\psi) \qquad 2.16$$

The boundary conditions for a cantilever beam are, for all t,

at free end,

$$\{x=0;\ \psi=(\gamma H)^a/_b\}$$

$$\frac{\partial y}{\partial x}=0;\ \frac{\partial y}{\partial \psi}=0 \qquad 2.17a$$

at built in end

$$\{x=H;\ \psi=\gamma H\,(1+{}^a/_b)\}$$

$$y=0 \qquad 2.17b$$

Using boundary conditions and solving for arbitrary constants D_1' and D_2', a frequency equation is obtained which is of the form

$$\tan \gamma H = -\gamma H \left(\frac{a}{b}\right) \qquad 2.18$$

This is a transcendental equation and can be solved graphically.

For other cases of non-uniform beams, not covered by equation 2.10, the solution is cumbersome and numerical methods may be adopted.

3. Numerical Solution

If the variation of area f(x) along the length of the beam is a complicated function, then the theoretical solution of equation 1.4 is practically ruled out. It is therefore desirable to adopt numerical techniques in solving such problems. If, however, a high speed computer is available, even simpler problems are better solved by numerical methods.

The method consists in replacing a continuous system with a discrete system by concentrating the mass distribution into an equivalent set of discrete point masses embedded in an ideal massless substance possessing the same elastic properties as the body simulated.

ERROR ANALYSIS OF THE NUMERICAL APPROACH

Errors are due to approximating an infinite degree of freedom system to a finite degree of freedom system and not due to the numerical technique involved. One type of error involves the number of masses used. The other type involves the determination of equivalent masses and stiffnesses.

To make an error analysis one should know the exact value of items under investigation. The errors in frequency would be investigated here for uniform cantilever shear beams for which exact theoretical solutions are available.

ERRORS IN FINDING EQUIVALENT MASSES

There are generally two procedures adopted in finding out equivalent masses. In one case, the mass of a segment is divided equally and concentrated at the ends (Fig. 3.1). In the other case, the mass is concentrated at the centre of a segment (Fig 3.2).

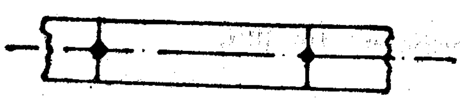


FIG 3.1

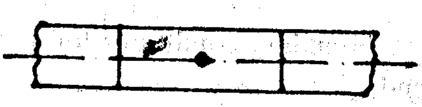


FIG 3.2

It has been shown (Duncan—1952) that for a uniform shear beam, if the mass points are located at midpoints of equal segments, the error in frequency varies inversely as the square of number of segments whereas if the masses are placed at the ends, the error varies as inverse first power of number of segments.

ERRORS DUE TO NUMBER OF SEGMENTS

In the case of shear beam in addition to having an exact solution for the continuous system, we also have an exact solution for the discrete system (Karman and Biot—1940).

For the continuous system, frequency parameter

$$\lambda_r = \frac{(2r-1)^2 \cdot \pi^2}{4} \tag{3.1}$$

(λ is proportional to p^2)

For the discrete case, with mass points concentrated at the middle of segments

$$\lambda_{rn} = 2n^2 \left[1 - \cos \frac{(2r-1)\,\pi}{2n} \right] \tag{3.2}$$

$$= \frac{(2r-1)^2\,\pi^2}{4} - \frac{(2r-1)^4 \cdot \pi^4}{192n^2} + \text{small terms} \tag{3.2a}$$

Here n is the number of segments

$$\text{Error } \varepsilon_{rn} = \frac{\lambda_r - \lambda_{rn}}{\lambda_r} \tag{3.3}$$

$$= \frac{(2r-1)^2 \cdot \pi^2}{48n^2} + \text{higher inverse powers of n} \tag{3.3a}$$

This shows that the error in λ (that is, p^2) for any given mode ultimately varies inversely as the square of number of segments and that proportional error for a given n increases rapidly for higher harmonics. Fig. 3.3 shows a plot of ε_{rn} versus number of segments for the first four modes of vibration.

In the problems attempted by numerical method, n was chosen as 100. Even if n had been chosen as 40, it is seen that error would be negligible.

In the case of non-uniform beams, if the mass is assumed to be concentrated at the centre of gravity of the segments instead of at the middle of the segments, results obtained are extremely close to the exact values. (error in frequencies is less than 0.25% in all cases considered for this investigation). Even for the assumption that the mass is concentrated at the middle of the segments, the error in frequencies was less than 0.85% in all cases considered for this investigation.

HOLZER METHOD

This technique is very suitable to solve equations of this type numerically. Consider equations 1.1 and 1.2 and assume that a harmonic solution in time with frequency p is applicable. Then, for the free vibration problem

$$V = -\sigma' AG \frac{\partial y}{\partial x} \tag{3.4}$$

$$\frac{dV}{dx} = mp^2 y \tag{3.5}$$

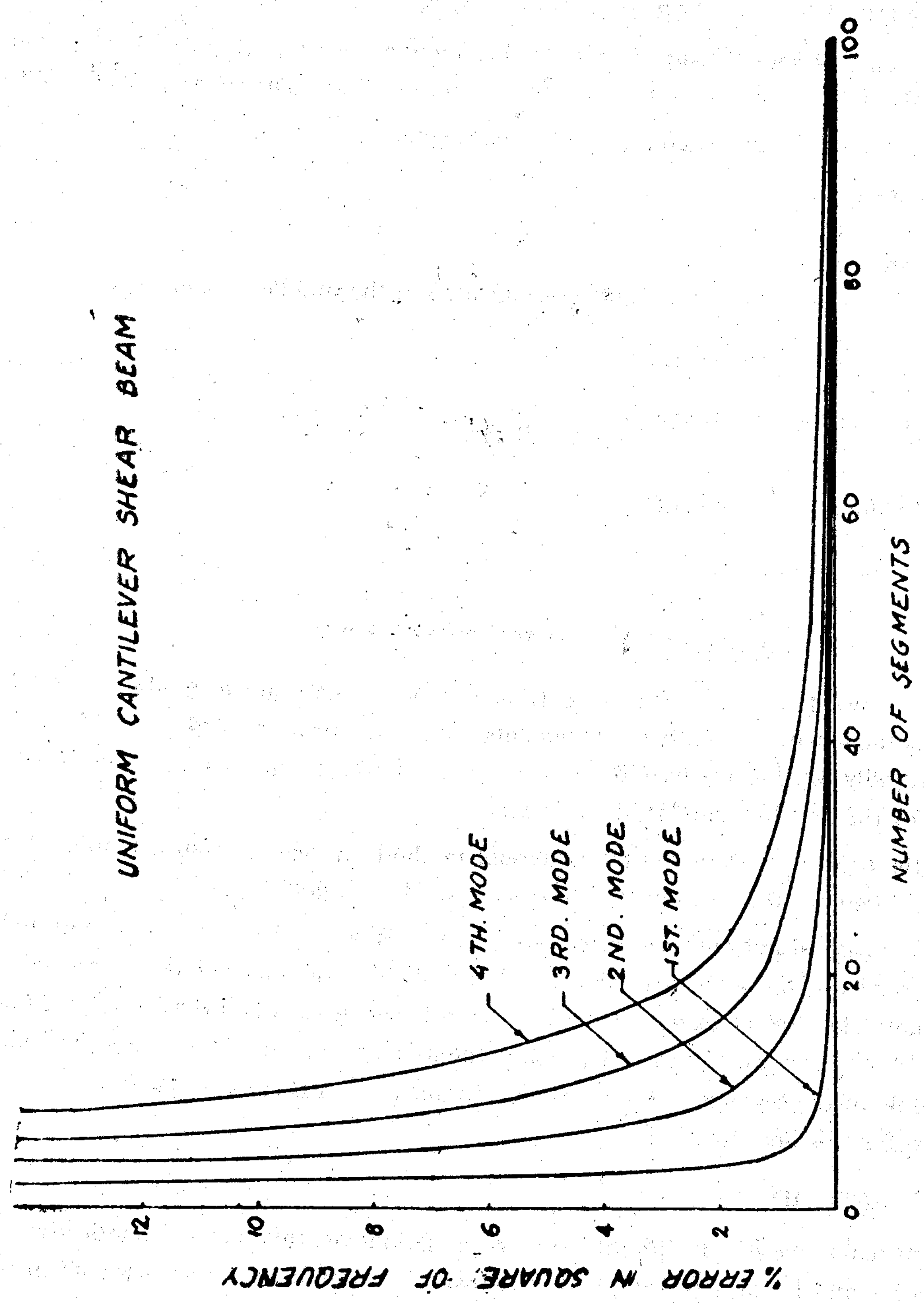


Figure 3.3

Let the beam be divided into a number of equal segments and one typical section of beam be as shown in fig. 3.4.

A finite change of shear force occurs at each mass which is equal to inertia force of mass

$$\triangle V = mp^2 y \qquad 3.6$$

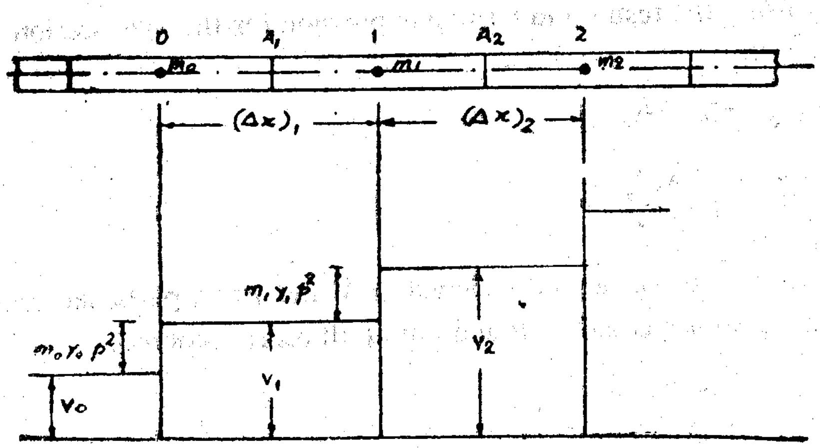


Figure 3.4

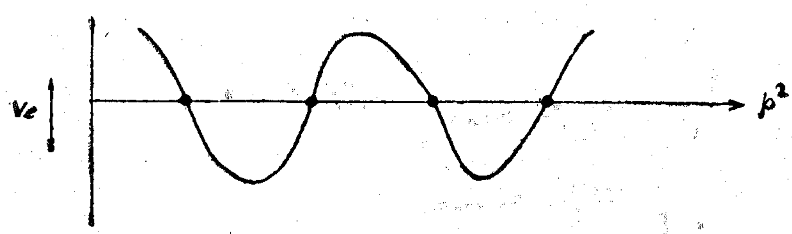


Figure 3.5

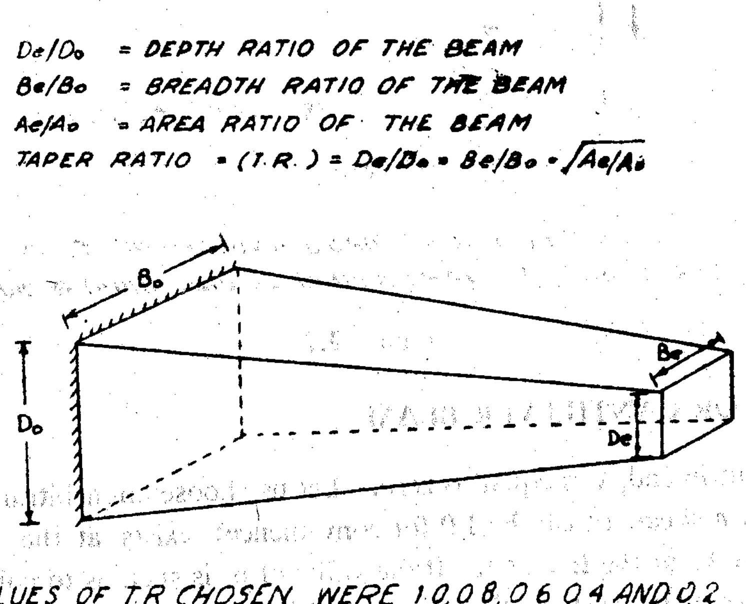


Figure 3.6

Assume that the quantities V_0 and y_0 are known at the left section. Then

$$V_1 = V_0 + m_0 p^2 y_0 \tag{3.7}$$

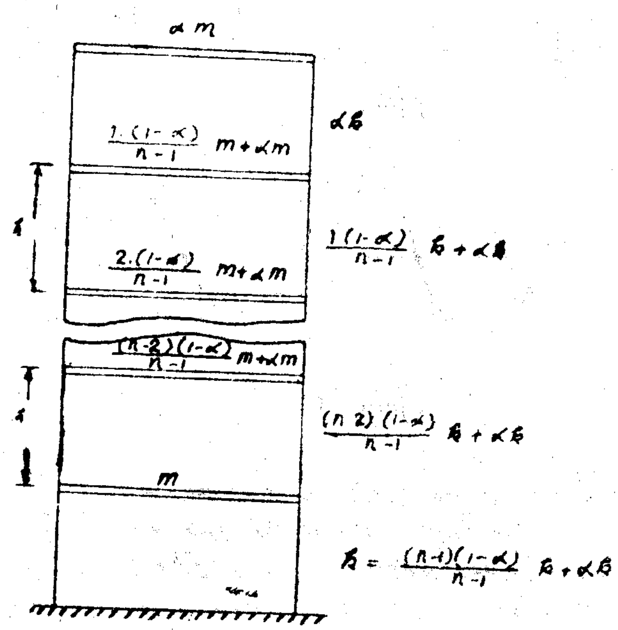
$$y_1 = y_0 - \left(\frac{V_1}{\sigma' A_1 G}\right)(\Delta x)_1 \tag{3.8}$$

Generalising the result and writing expression for the n^{th} section in terms of values at $(n-1)^{th}$ section

$$V_n = V_{n-1} + m_{n-1} p^2 y_{n-1} \qquad 3.9$$

$$y_n = y_{n-1} - \left(\frac{\Delta x}{\sigma' AG}\right)_n V_n \qquad 3.10$$

Thus, for any frequency p, if the values V and y at a particular section is known, then the corresponding values could be found out at all other sections.



BUILDING MODEL WHERE MASS AND SPRING CONSTANT ARE ASSUMED TO VARY LINEARLY OVER HEIGHT OF BUILDING.

Figure 3.7

PROCEDURE FOR CANTILEVER BEAM

At the built in end, y is equal to zero. Let us choose an arbitrary value for p, say p′. First assume that a shear (it can be 1.0 for convenience) exists at the built-in-end and then evaluate the shear V_e at the free end. If the value of p′ is such as to coincide with one of the natural frequencies of the system, then V_e should be zero.

In general, it would not be possible to guess the value of p correctly. However, various values could be arbitrarily assigned to p and then V_e evaluated. A plot of p^2 versus V_e would have a general appearance as in fig. 3.5. The correct value of p^2 are those which correspond to the intersection of the curve with the p^2 axis.

4. Specification of the Problems

Linearly tapering beams of the type shown in fig. 3.6 have been considered for dynamic analysis.

In all cases, the frequencies and responses of the beam like relative displacement with respect to the base, shear and moment at all sections due to ground motion have been calculated by the mumerical method outlined above.

5. Discussion of Results

In practice; no beam would have predominantly shear deformations. However, shear beams could be theoretical models of multistoreyed framed structures as the equations of motion governing the behaviour of the two systems are analogus.

UNIFORM CASE

For a uniform multistoreyed framed structure, the frequency of vibration in the r^{th} mode is (Chandrasekaran–1963)

$$p_r = 2\sqrt{k/m}\,.\,\sin\frac{2r-1}{2n+1}\,.\,\pi/2 \qquad 5.1$$

For a uniform shear beam,

the mass of the beam $= \rho\, AH$ 5.2

considering the beam to be divided into n equal parts

equivalent mass of element $= \dfrac{\rho\, AH}{n}$ 5.3

the spring constant of the beam $= \dfrac{A\sigma' G}{H}$ 5.4

equivalent spring constant element$= \dfrac{nA\,\sigma' G}{H}$ 5.5

Substituting 5.3 and 5.5 in 5.1, and observing that as n is very large for the shear beam,

$$\sin\frac{2r-1}{2n+1}\,\frac{\pi}{2} \approx \frac{2r-1}{2n+1}\,\pi/2, \qquad \text{then}$$

$$p_r = 2\sqrt{\frac{nA\sigma' G}{H} \times \frac{n}{\rho A H}} \times \frac{2r-1}{2n+1}\,\pi/2 \qquad 5.6$$

$$= \frac{(2r-1)}{2} \times \frac{\pi}{H} \times \sqrt{\frac{\sigma' G}{\rho}} \qquad 5.6a$$

Equation 5.6a is same as equation 2.7

NON-UNIFORM CASE

From 5.3, the variation of mass along the height for M.S.F.S. (multistoreyed framed structure) should correspond to variation of ρA along the height for C.S.B. (cantilever shear beam). Similarly, from 5.5, the variation of spring constant along the height for M.S.F.S. should correspond to variation of σ' GA along the height for a C.S.B.

Consider building models for which the mass and stiffness varies linearly along the height and the linear variation is represented by parameter α (Refer fig. 3.7). If it is assumed that for an equivalent shear beam ρ and σ' G remain constant over the height of the beam, then the variation of A should correspond to α. Since the variation of A is represented by $(T.R.)^2$, (refer fig. 3.6), α corresdonds to $(T.R.)^2$.

COMPARISON OF M.S.F.S. AND C.S.B. FOR LINEARLY TAPERING CASES

Dynamic analysis has been carried out for M.S.F.S. (Chandrasekaran–1963). The natural frequencies of vibration and responses of the system like relative displacement with respect to the base, shear and moment at all sections due to ground motion have been calculated.

Also, dynamic analysis of C.S.B. has been carried out for various taper ratios. The various quantities could be expressed as follows:—

	M.S.F.S.	C.S.B.
Natural frequency in the r^{th} mode of vibration, p_r	$C_1^{(r)}.\sqrt{\frac{k}{m}}$	$C'^{(r)}_1.\frac{1}{H}.\sqrt{\frac{\sigma' G}{E}}.\sqrt{\frac{E}{\rho}}$
Relative Displacement at any section i, in the r^{th} mode of vibration, $Z_i^{(r)}$	$C_{21}^{(r)}.\sqrt{\frac{m}{k}}.S_v$	$C'^{(r)}_{21}.\sqrt{\frac{E}{\sigma' G}}.\sqrt{\frac{\rho}{E}}.H.S_v$
Shear at any section i in the r^{th} mode of vibration, $V_i^{(r)}$	$C_{31}^{(r)}.\sqrt{km}.S_v$	$C'^{(r)}_{31}.\sqrt{\frac{\sigma' G}{E}}.\sqrt{\frac{\rho}{E}}.EA.S_v$
Moment at any section i in the r^{th} mode of vibration, $MT_i^{(r)}$	$C_{41}^{(r)}.\sqrt{km}.h.S_v$	$C'^{(r)}_{41}.\sqrt{\frac{\sigma' G}{E}}.\sqrt{\frac{\rho}{E}}.EA.HS_v$

The following quantities of M.S.F.S. (multistoreyed framed structure) and C.S.B. (cantilever shear beam) are analogous to each other.

M.S.F.S.	C.S.B.
m	$\frac{\rho AH}{n}$
k	$\frac{n\sigma' AG}{H}$
α	$(T.R.)^2$

Figures 5.1 to 5.4 show respectively plots of p, Z, V and MT versus α. In all the cases, it is observed that a C.S.B. could be an analogous theoretical model of M.S.F.S.

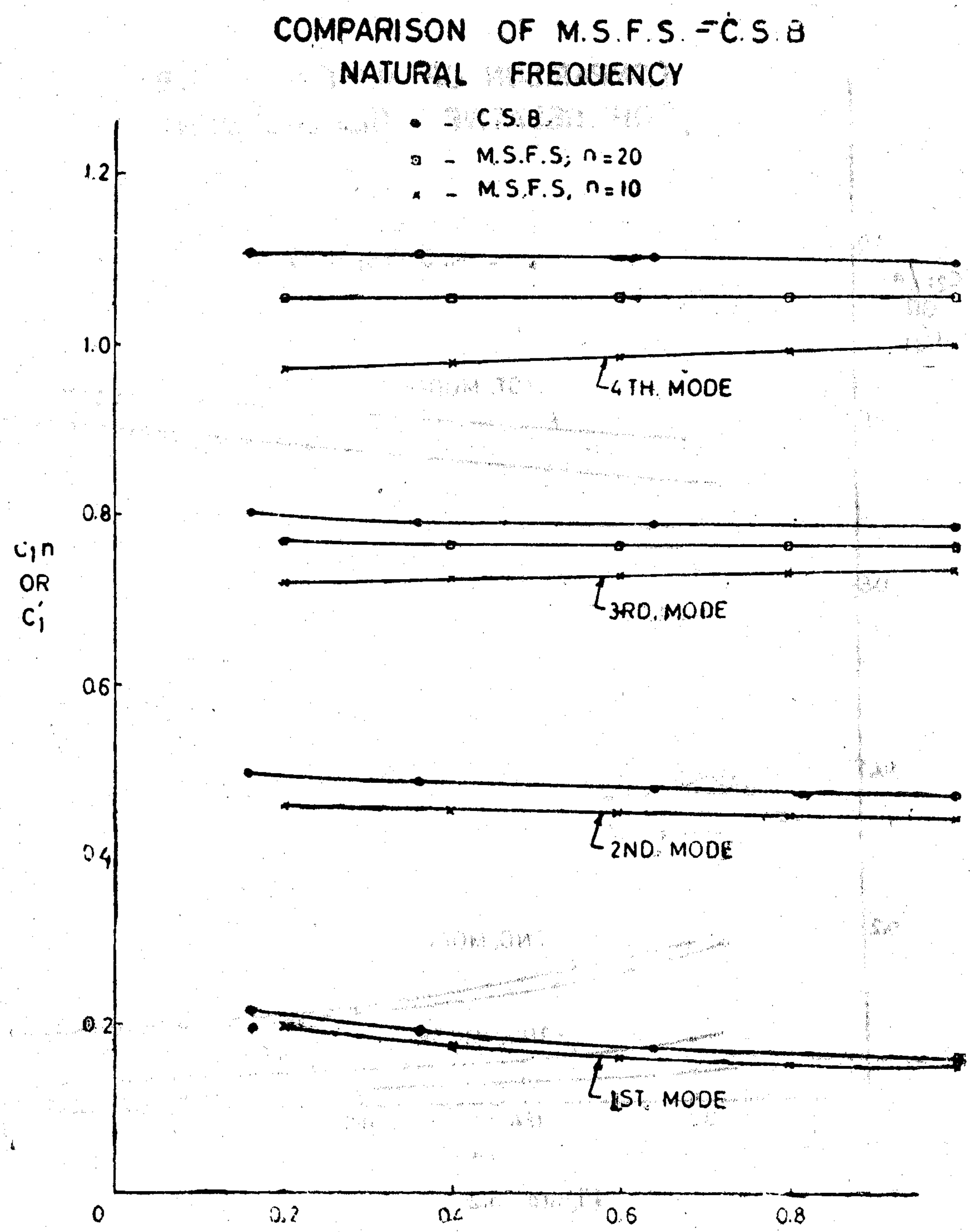


Figure 5.1

Figures 5.5 to 5.9 show plots of shear diagram. It is also observed that shear diagram for a C.S.B. is very close and similar to that of a corresponding shear diagram of M.S.F.S.

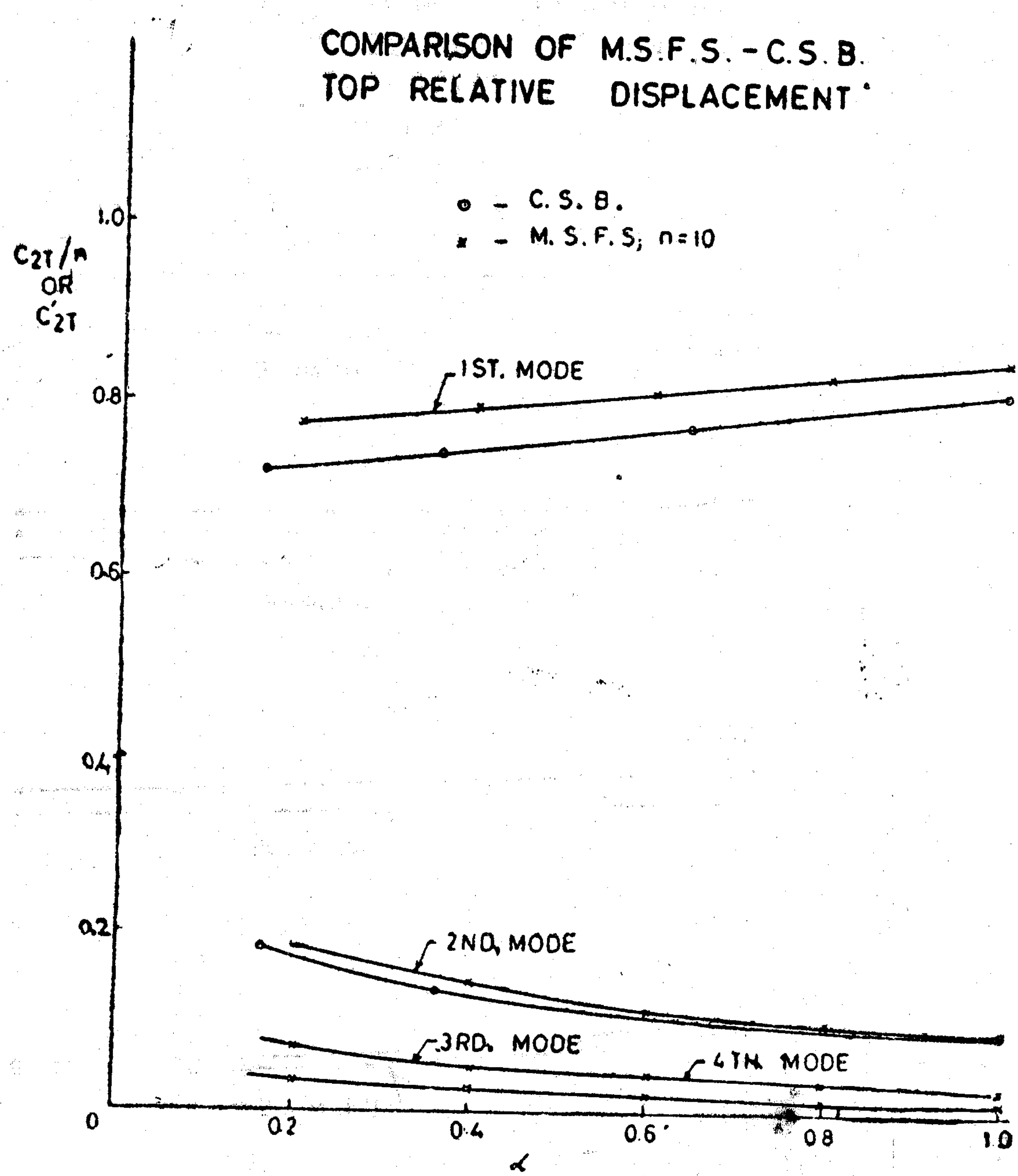

COMPARISON OF M.S.F.S. - C.S.B.
TOP RELATIVE DISPLACEMENT
o - C. S. B.
x - M. S. F. S; n=10
C2T/n
OR
C'2T
1.0
0.8
0.6
0.4
0.2
0
1ST. MODE
2ND. MODE
3RD. MODE
4TH. MODE
0.2
0.4
0.6
0.8
1.0
α

Figure 5.2

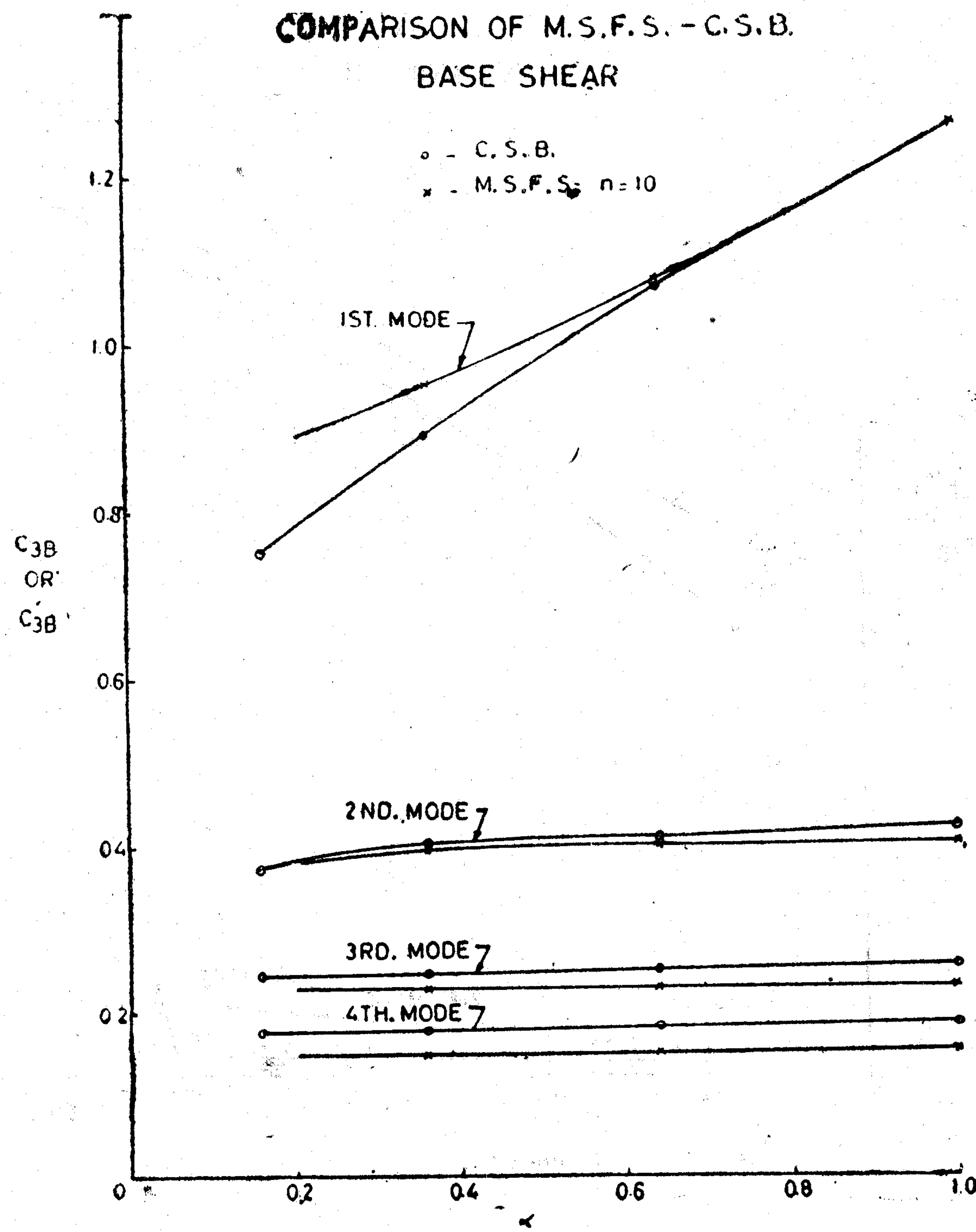

COMPARISON OF M.S.F.S. - C.S.B.
BASE SHEAR
o - C.S.B.
x - M.S.F.S. n=10
1ST. MODE
2ND. MODE
3RD. MODE
4TH. MODE
C3B OR C'3B
1.2
1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1.0

Figure 5.3

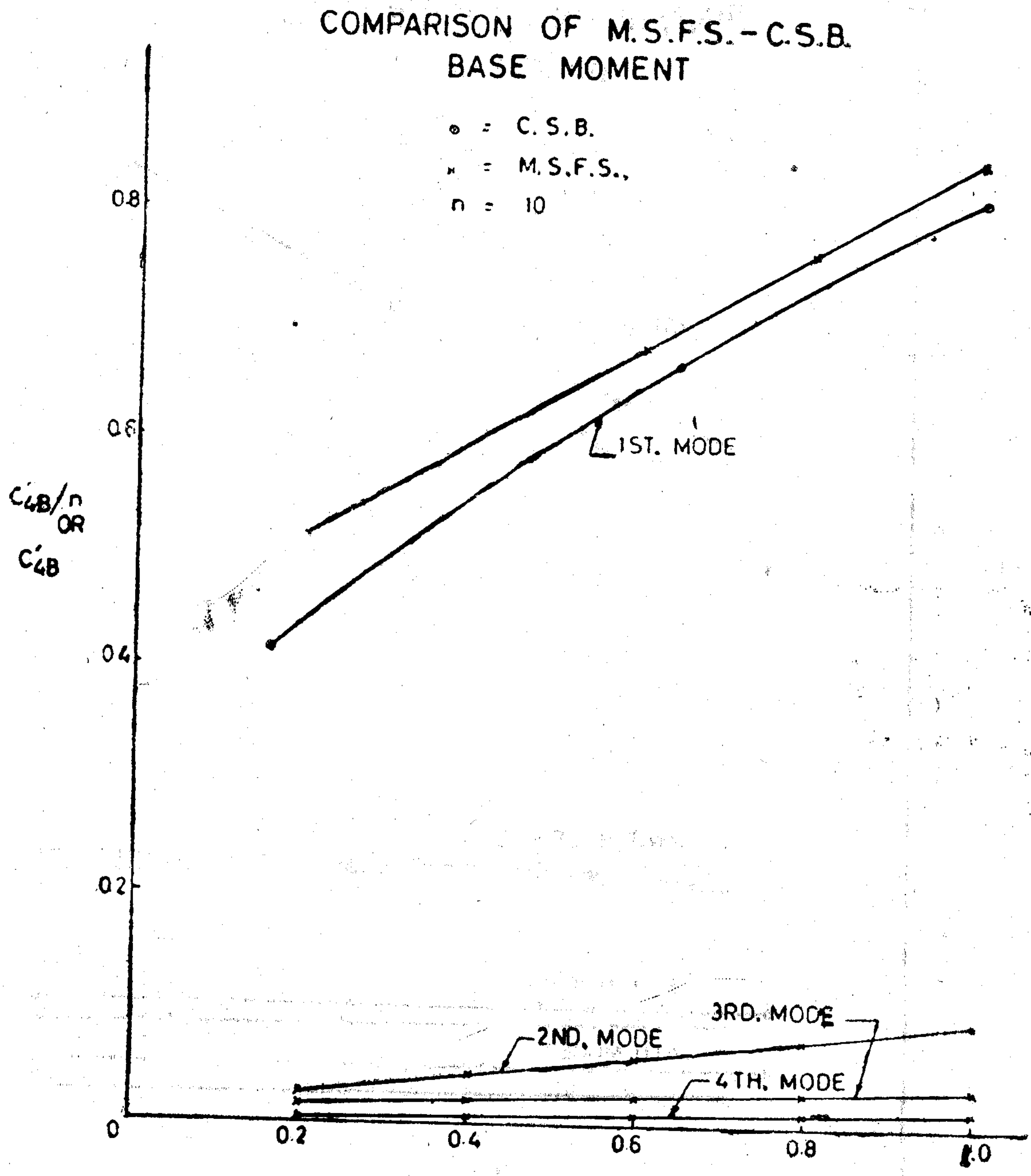

COMPARISON OF M.S.F.S.-C.S.B.
BASE MOMENT
⊙ = C.S.B.
× = M.S.F.S.,
n = 10
0.8
0.6
0.4
0.2
0
C'4B/n
OR
C'4B
1ST. MODE
2ND. MODE
3RD. MODE
4TH. MODE
0.2
0.4
0.6
0.8
1.0

Figure 5.4

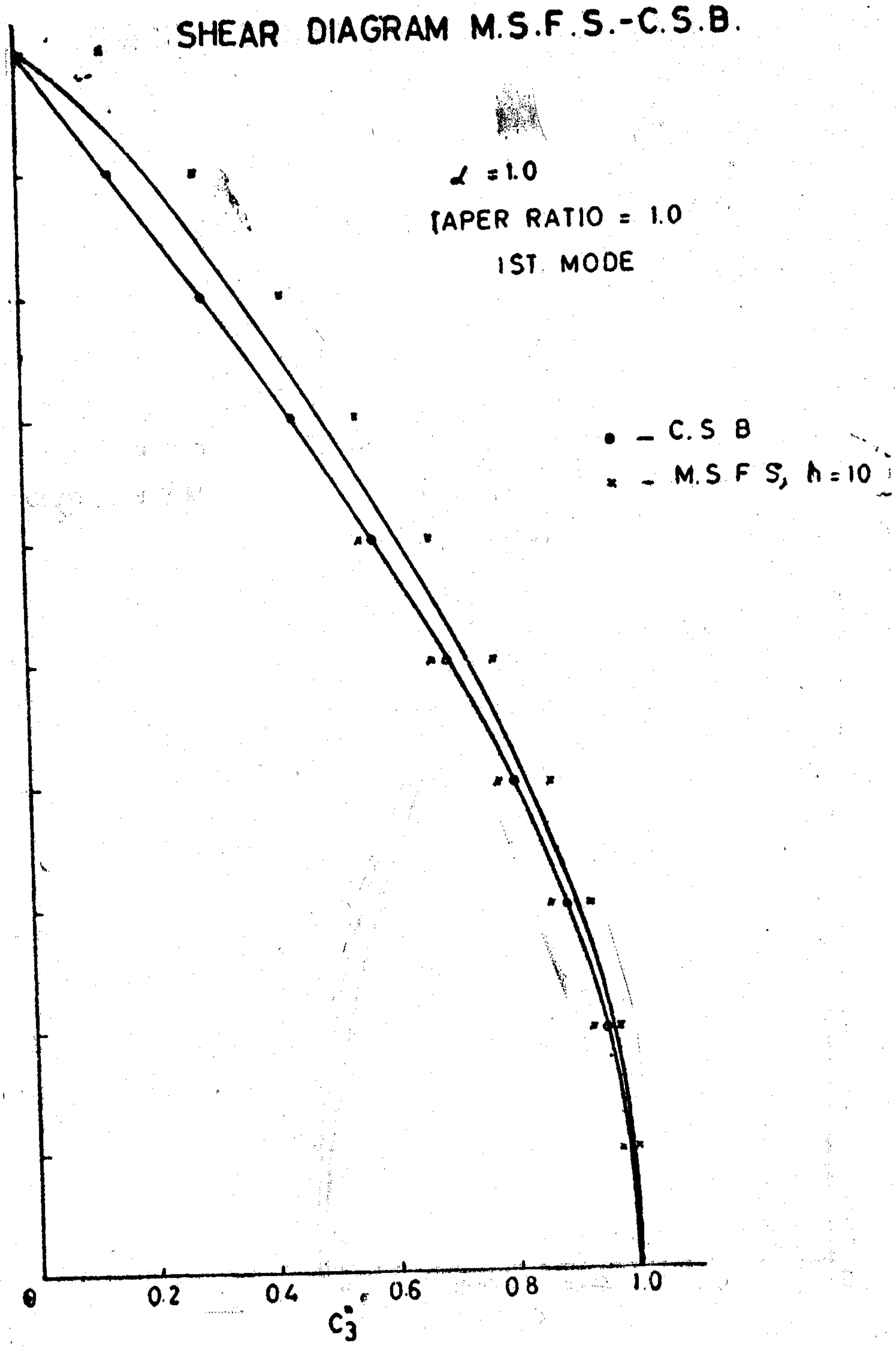

SHEAR DIAGRAM M.S.F.S.-C.S.B.
α = 1.0
TAPER RATIO = 1.0
1ST MODE
• – C.S B
× – M.S F S, h = 10
0
0.2
0.4
0.6
0.8
1.0
C3*

Figure 5.5

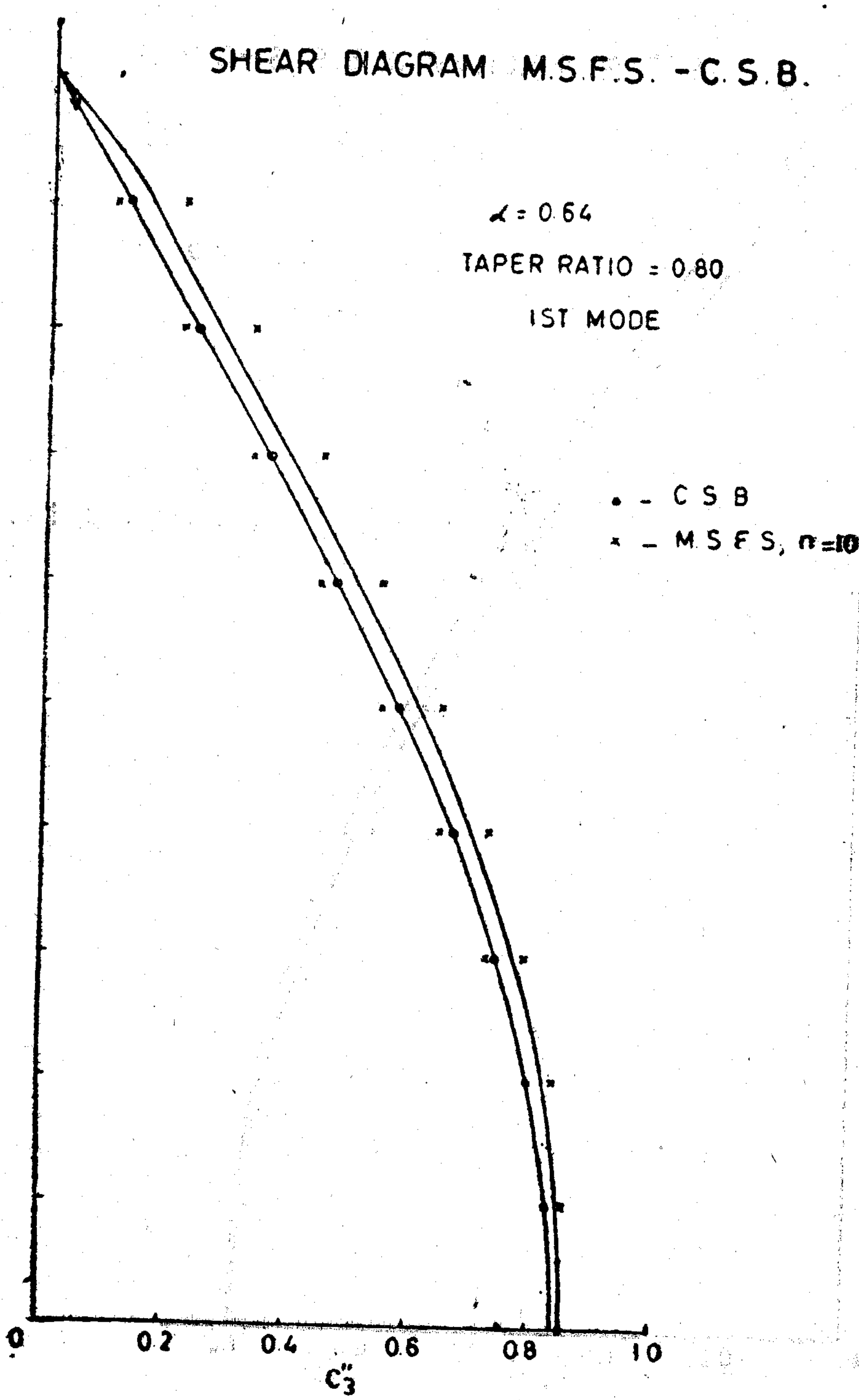

SHEAR DIAGRAM M.S.F.S. - C.S.B.
α = 0.64
TAPER RATIO = 0.80
IST MODE
• - C S B
× - M S F S, n=10
0
0.2
0.4
0.6
0.8
1.0
C''3

Figure 5.6

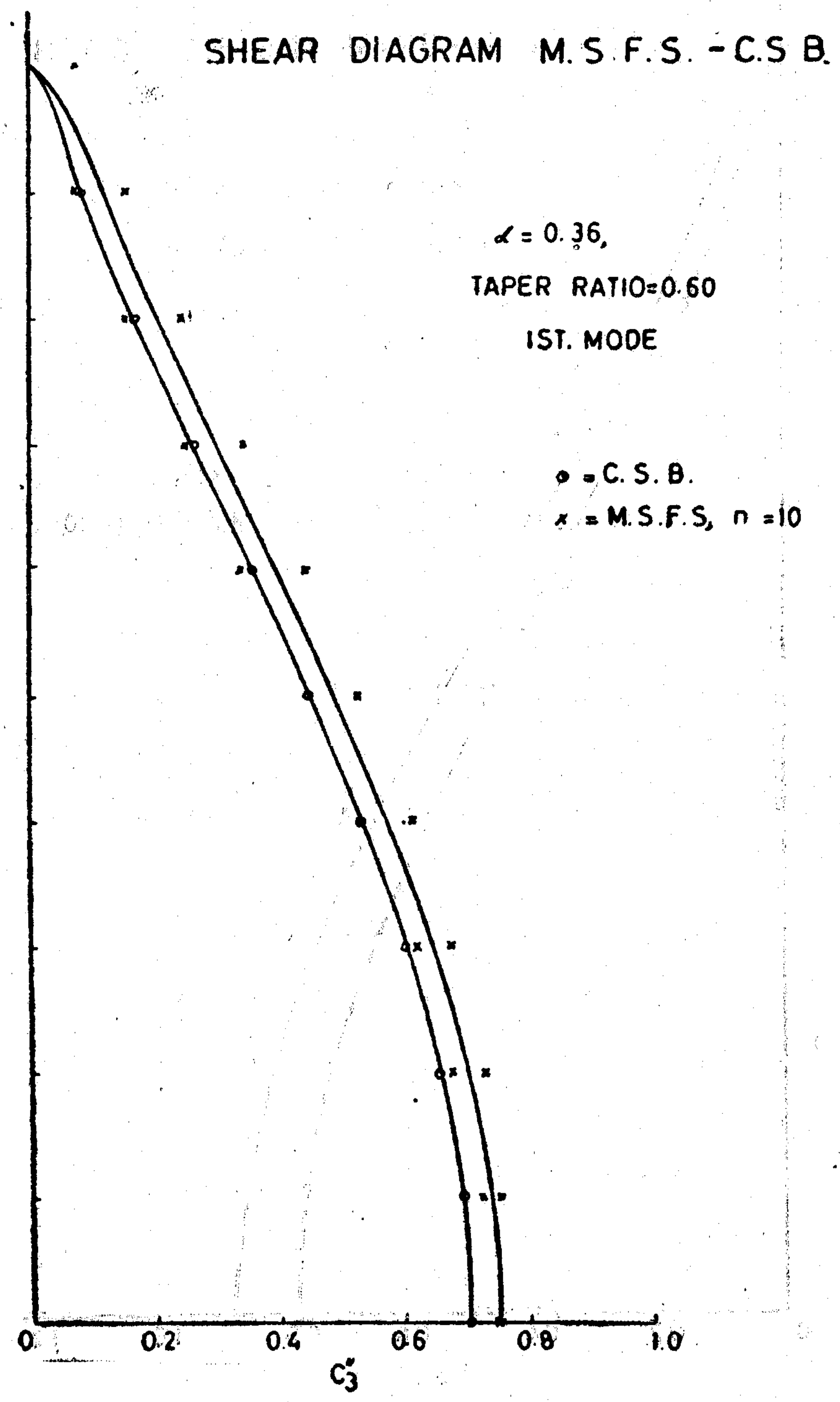

SHEAR DIAGRAM M. S. F. S. - C. S. B.
α = 0.36,
TAPER RATIO=0.60
1ST. MODE
o = C. S. B.
x = M. S. F. S, n = 10
0
0.2
0.4
0.6
0.8
1.0
C3"

Figure 5.7

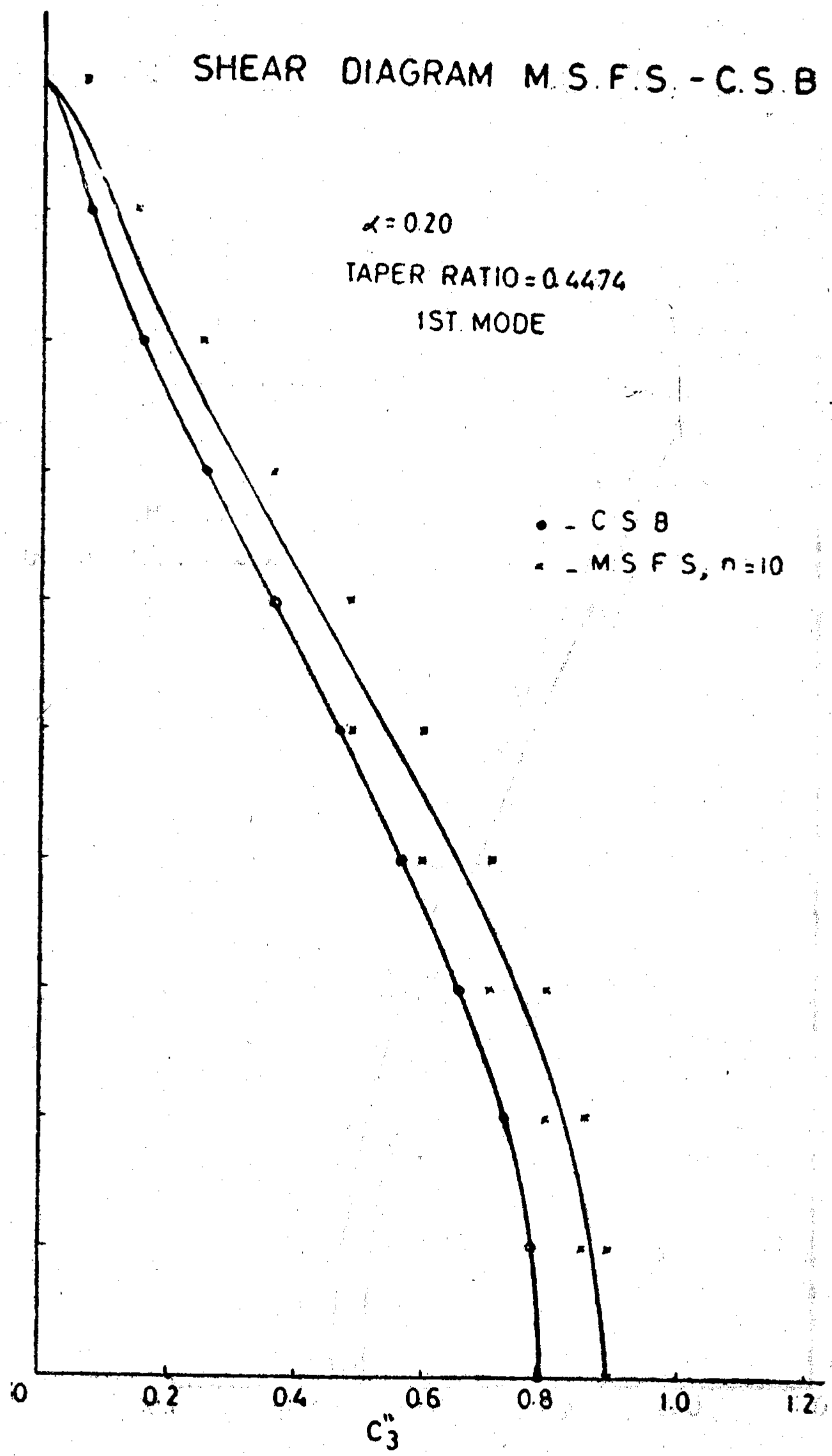

SHEAR DIAGRAM M.S.F.S. - C.S.B
α = 0.20
TAPER RATIO = 0.4474
1ST MODE
• - C S B
× - M S F S, n = 10
0
0.2
0.4
0.6
0.8
1.0
1.2
C"3

Figure 5.8

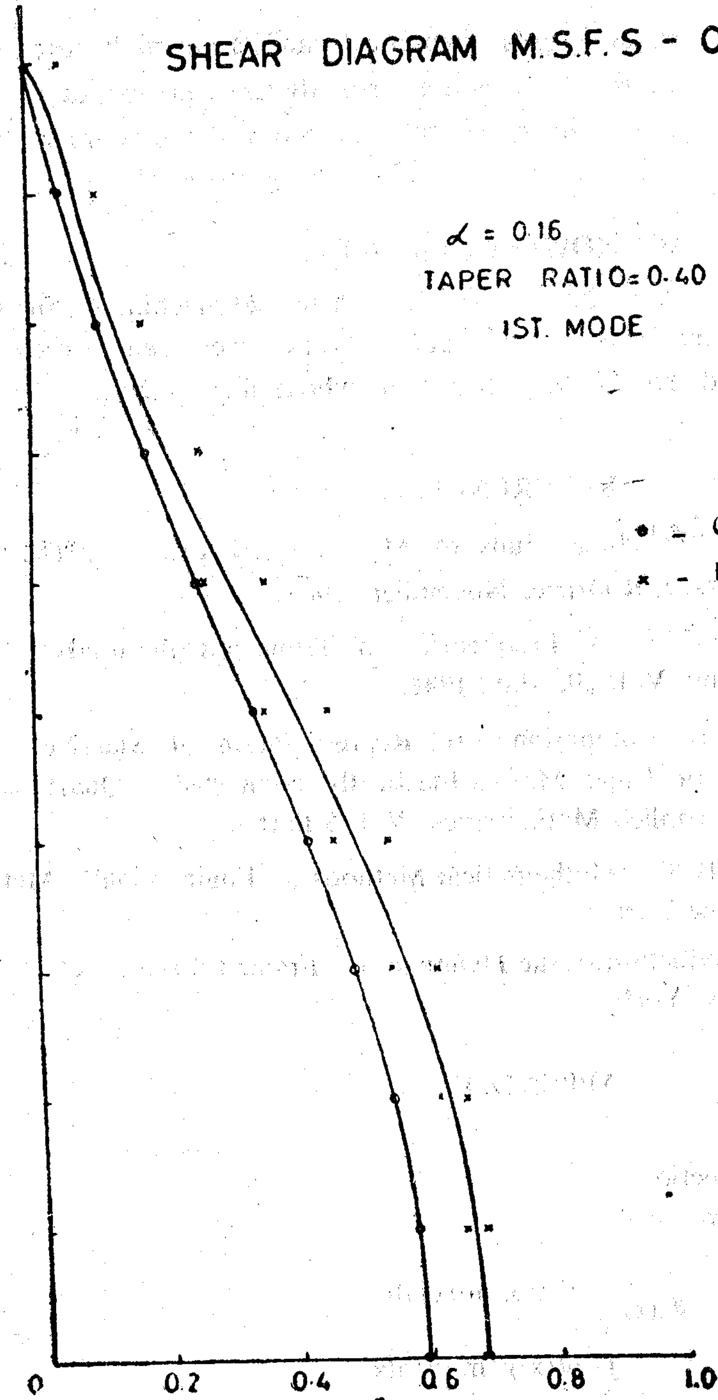

SHEAR DIAGRAM M.S.F.S - C.S.B.
α = 0.16
TAPER RATIO = 0.40
IST. MODE
• – C.S.B.
× – M.S.F.S, n = 10
0
0.2
0.4
0.6
0.8
1.0
C"3

Figure 5.9

CONCLUSION

Cantilever shear beams are analogous theoretical models of multistoreyed framed structure. The concept of shear beam indicates that once dynamic properties are calculated for a multistoreyed framed structure (say for n=5), then the behaviour of similar multistoreyed structures (that is for all values of n above 5) could be easily predicted.

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APPENDIX

Notations

A — Area of cross section

a(t) — Ground acceleration

B_x — Mode factor $= \dfrac{\phi(x) \int_0^H \phi(x).\, m(x)\, dx}{\int_0^H (\phi(x))^2\, m(x)\, dx}$

C_1, C_1' — Frequency coefficients

C_2, C_2' — Displacement coefficients; subscript 'T' used along with this represents values at top

C_3, C_3' — Shear coefficients; subscript 'B' used along with this represents values at base

C_3'' — $C_3'/1.2732$

C_4, C_4' — Moment coefficients; subscript 'B' used along this represents values at base
D — arbitrary constant to be determined from initial or boundary conditions
E — modulus of elasticity
G — modulus of rigidity
H — Total length of beam
k — spring constant
m — mass
MT — moment at any section
n — number of masses
p — natural frequency of system
p_d — damped natural frequency of system; $\approx$ p if ζ is small, say, $\leqslant 0.20$
r — index representing mode of vibration
S_v — Response velocity spectrum

$$= \left| \int_0^t a(t) . e^{-\zeta p (t-\tau)} \sin p_d (t-\tau) \, d\tau \right|_{\text{maximum}}$$

t — time interval
T.R. — Taper Ratio, defined as per fig. 3.6
V — shear force at any section
w — intensity of load at any section
x — distance measured along the length of the beam
y — displacements measured transverse to the longitudinal axis of the beam
Z — relative displacement of any section with respect to the base
α — mass and spring constant variation parameter defined as per figure 3.7
ζ — coefficient of damping expressed as a fraction of critical damping value
θ — phase angle between input and output
ξ — normal coordinate
ρ — mass density of beam
σ' — ratio of average shear stress on a section to the product of shear modulus and angle of shear at the neutral axis.
ϕ — mode shape coefficient