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ON THE USE OF NUMERICAL INTEGRATION TO SOLVE SEISMIC WAVE PROPAGATION PROBLEMS

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INTRODUCTION

A major problem before seismologists, earthquake engineers and exploration geophysicists engaged in seismic exploration is how to synthesize seismograms for realistic earth models. We report here one of our experiences in attempting such problems by evaluating the integrals of the formal solutions numerically.

The solution to the problem of generating synthetic seismograms is required by seismologists to interpret the observed seismograms for deep earth structure and the earthquake source and by exploration geophysicists for shallow earth structures in areas likely to hold economic deposits of minerals, notably petroleum. Earthquake engineers are interested in synthetic seismograms for a number of reasons, chief among which is the need to predict ground motion to be expected over realistic earth models in response to earthquake induced excitation so that suitable safety provisions may be incorporated economically in the design of important man-made structures. Although the recent trend in earthquake engineering is to represent observed ground motion using statistical models, deterministic synthetic seismograms should provide useful constraints. In support of this remark we cite the observation that strong ground displacements computed as functions of time from observed accelerograms (see for example, Housner, 1970) are reasonably smooth.

The first synthetic seismograms were produced by Lamb (1904) when he obtained the solution to the problem of transient line and point explosive sources on the surface of an elastic solid half space, a highly simplified model of the earthquake source as well as the earth. Lamb solved the boundary-value initial-value problem using two Fourier transforms, one in a spatial coordinate and another in time. The difficulties arose in evaluating the inverse Fourier transforms after the initial and boundary conditions had been applied. Lamb solved these integrals approximately using residue theory. Subsequently, Nakano (see Lapwood, 1949) and Lapwood (1949) examined the problem of buried sources and obtained synthetic seismograms from

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approximate analytical techniques. Garvin (1956) applied a variant of the Cagniard technique and obtained exact synthetic seismograms for the problem of a transient line source in a solid half space. More recently, a host of other attempts have been made to synthesize seismograms. But we mention here particularly the work of Kelly et al. (1976) who used the method of finite differences and Rao and Goda (1978) who used numerical integration for the evaluation of both the inverse transform integrals. The latter authors solved the problem of a point source in a solid half space as a prelude to attacking more complicated problems of transition layers in the half space. Their synthetic seismograms were not compared directly with seismograms computed in an alternative way to establish the validity of the method. We have examined the numerical integration idea also and compare in this paper such synthetic seismograms for the problem of a line source in solid half space with Garvin's exact solutions.

STATEMENT OF THE PROBLEM

Consider a cartesian coordinate system Oxyz. Let the free surface of an homogeneous, isotropic, perfectly elastic solid half space coincide with the xy plane and let the half space itself occupy the region of the positive z axis. Let a transient explosive line source pass through the point $(0, 0, h)$ and lie parallel to the y axis. Let the source vary with time as $h(t) \exp(-\eta t)$, where $h(t)$ is zero for t less than zero and unity there after and let η be a decay constant. Therefore, this is a two dimensional problem in which quantities vary spatially with x and z only, (Figure 1). The problem was solved formally by Lapwood (1949) who gave the x and z components of displacement at a surface point as

$$s_x(x, 0, t) = Re\left\{2j \int_0^\infty\int_0^\infty A\,(w^2+\eta^2)^{-1}\,[\eta \sin(wt-kx) - w\cos(wt-kx)]\,dk\,dw\right\};$$

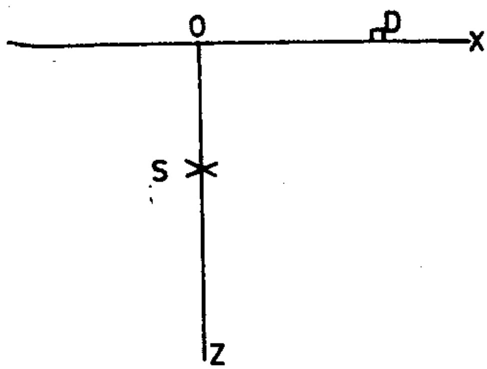

Fig. 1. The coordinate system and the model. y axis points normally out of the plane of the figure and passes through 0. The source passes through S and is parallel to y-axis.

$$s_z(x, 0, t) = Re\left\{ 2 \int_0^\infty \int_0^\infty B\,(w^2 + \eta^2)^{-1} [\eta \cos(wt - kx) + w \sin(wt - kx)]\, dk\, dw \right\}.$$

Here,
$$A = 4jkk_\beta^2 \nu' \exp(-\nu h)/F(k);$$
$$B = 2k^2\left(2k^2 - k_\beta^2\right) \exp(-\nu h)/F(k);$$
$$F(k) = \left(2k^2 - k_\beta^2\right)^2 - 4k^2 \nu\nu';$$
$$\nu' = \left(k^2 - k_\beta^2\right)^{\frac{1}{2}}$$
$$\nu = \left(k^2 - k_\alpha^2\right)^{\frac{1}{2}};$$
$$k_\alpha = w/\alpha;$$
$$k_\beta = w/\beta;$$
$$j = (-1)^{1/2};$$
w = angular frequency;
k = angular wave number;
α = compressional wave speed;
β = shear wave speed.

The reasons why these integrals cannot be evaluated immediately are that: (1) they are improper integrals; (2) the integrands are multi-valued because of branch points at $k = \pm k_\alpha$ and $k = \pm k_\beta$ in the k plane; and (3) the integrands have poles at values of k for which $F(k) = 0$ in the k plane and at $w = \pm j\eta$ in the w plane.

Our problem was to see if the integrals could be evaluated numerically.

NUMERICAL INTEGRATION

We first reduced the quantities in the above equations to dimensionless form. All lengths were expressed as multiples of $r = (x^2 + h^2)^{\frac{1}{2}}$ where x is the x coordinate of the observation point. Time was expressed in multiples of r/α. We represented the dimensionless variables by the same symbols as the corresponding dimensioned variables. Thus, $x = x/r$, $h = h/r$, $k = kr$, $s_x = s_x/r$, $s_z = s_z/r$ and $w = wr/\alpha$. Also, $\alpha = 1$ and $\beta = 0.5773505$ assuming that the medium is a Poisson solid.

We evaluated the integrals by replacing them by their Riemann sums with terms evaluated at discrete sampled values of w and k such that those values did not coincide with the singularities of the integrands. The actual expressions evaluated were

$$s_x(x, 0, t) = Re\left\{2j \sum_{l=1}^{N} \left(w_l^2 + \eta^2\right)^{-1} \sum_{p=1}^{M} A_{lp} [\eta \sin(w_l t - k_p x) - w_l \cos(w_l t - k_p x)] \Delta k \, \Delta w \right\};$$

$$s_z(x, 0, t) = Re\left\{2 \sum_{l=1}^{N} \left(w_l^2 + \eta^2\right)^{-1} \sum_{p=1}^{M} B_{lp} [\eta \cos(w_l t - k_p x) + w_l \sin(w_l t - k_p x)] \Delta k \, \Delta w \right\}.$$

Here A_{lp} and B_{lp} are values of A and B evaluated at w_l and k_p and Δw and Δk are sampling intervals in w and k planes. We note in passing that a total of NM terms were added for each x and t.

Following Rao and Goda (1978) values of w_1, w_N and Δw were chosen so as to satisfy the requirements of the discrete Fourier transform (DFT) (Bracewell, 1978). Accordingly, a synthetic seismogram, whether for s_x or s_z, was assumed to be cyclic with period T_s. Its digitized version was computed for 2^{2n+1} (n an integer) regularly spaced time intervals in each cycle. Thus the following relations were fixed automatically.

$$\Delta T_s \text{ (i.e., digitization interval)} = T_s/2^{2n+1};$$

$$w_1 = 2\pi/T_s;$$

$$w_N = w_{\text{Nyquist}} = \pi/\Delta T_s;$$

$$\Delta w = w_1;$$

$$N = w_{\text{Nyquist}}/\Delta w = 2^{2n}.$$

w_1, w_N and Δw were therefore all real. But, as a result, the k summation for each w_l ($i = 1, 2, \ldots, N$) had to be performed along a line parallel to the k_r (i.e., real k) axis a small distance ϵ above it so as to avoid the P and S branch points at $k = w_l$ and $k = 1.73\, w_l$ respectively and the Rayleigh pole at $k = 1.8839\, w_l$ (see Figure 2, contour 2). The following relations were assumed for each i.

$$k_1 = 0.95\, w_l + j\epsilon;$$

$$k_M = m w_l + j\epsilon, \quad m \text{ a real number greater than } 0.95;$$

$$\Delta k = (k_M - k_1)/(M - 1) = (m - 0.95)\, w_l/(M - 1),$$

The value of m was chosen by trial and error such that the resulting seismograms for m and $m + \Delta m$ were negligibly different. The values of w and k_r sampled according to this scheme fell on a grid as shown in Figure 3.

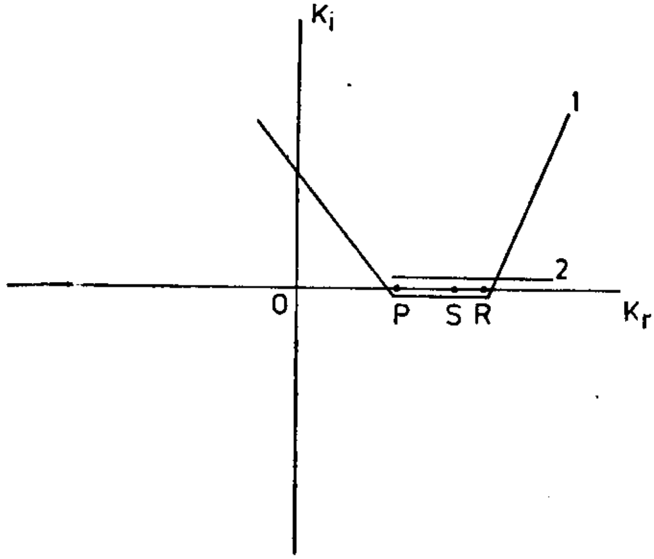

Fig. 2. Contour of integration in the k plane. Contour 1 used by Rao and Goda (1978). Contour 2 used in this study. P and S are branch points and R is the Rayleigh pole.

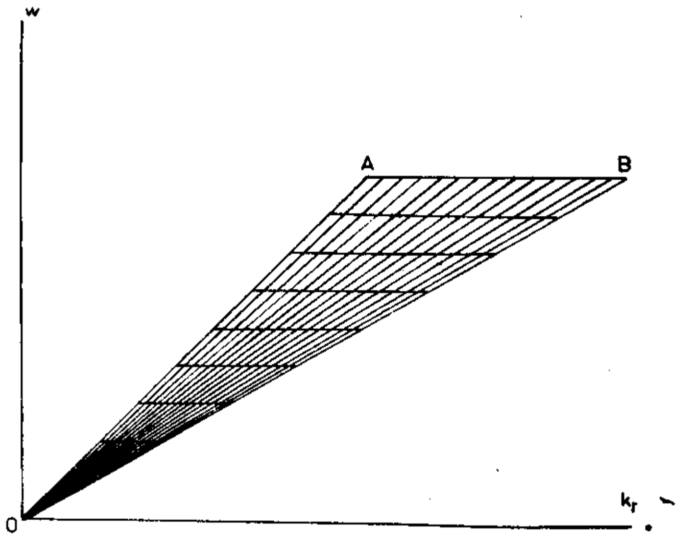

Fig. 3. Grid of sampled values in w and k_r. The figure is schematic.

RESULTS AND DISCUSSION

Traces 1 and 3 in Figure 4 represent synthetic seismograms for s_z and s_x computed according to the scheme discussed above for numerical integration, while traces 2 and 4 are exact results according to Garvin's solution of the same boundary-value initial-value problem (Bhandari, 1981). The synthetic seismograms of traces 1 and 3 were computed for $T_s = 2.56$ in dimensionless time, a reasonably long interval relative to the interval 0.8839 between the P and Rayleigh pulses. The following were the numerical values of other variables during this computation.

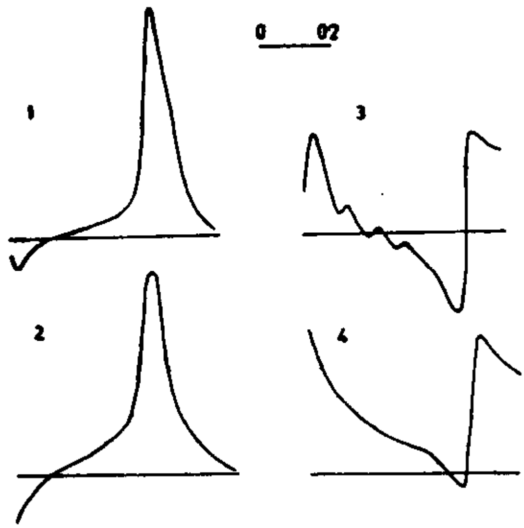

Fig. 4. Synthetic seismograms. Traces 1 and 3 were computed during this study. Traces 2 and 4 were computed by Bhandari (1981) after Garvin (1956). Scale of dimensionless time on the horizontal axis is indicated. The vertical scale is arbitrary. Traces 1 and 2 refer to s_z and 3 and 4 to s_x.

$$n = 2$$

$$T_s = 2.56/2^5 = 0.08;$$

$$w_1 = 2\pi/2.56 = 2.4544;$$

$$w_N = \pi/0.08 = 39.2699;$$

$$\Delta w = 2.4544;$$

$$N = 16;$$

$$\epsilon = 0.01;$$

$$k_1 = 0.95\ w_i + 0.01\ j, \quad \text{for each } i;$$

$$m = 2.85;$$

$$k_M = 2.85\ w_i + 0.01 j, \quad \text{for each } i;$$

$$M = 128;$$

$$\eta = 0;$$

$$x = 10;$$

$$h = 0.5.$$

The synthetic seismograms emphasize the P and Rayleigh pulses. There is no S pulse as such because we have an explosive source and a shear wave can only be expected as a result of mode conversion when the P pulse interacts with the free boundary of the medium. Such a converted pulse would be a part of the Rayleigh pulse. The particle motion in the synthesized Rayleigh pulse is retrograde elliptical as expected from other theoretical considerations.

Comparison of traces 1 and 2 and traces 3 and 4 leads us to the conclusion that there is a qualitative agreement between the numerically integrated and exactly computed traces. However, detailed inspection of the traces reveals important differences, most notably the presence of spurious oscillations on trace 3.

Traces 1 and 3 are the result of extensive testing in regard to values of u, m, M and ϵ. The rule of thumb that emerged was that the magnitudes of these quantities should be increased gradually until the shape of the seismogram stabilizes. We speculate that the spurious oscillations of trace 3 could be suppressed if the number NM of terms to be added is made very large. The speculation is buttressed by the fact of the well-known slow convergence of the Fourier series. The speculation, however, was not tested on the computer because of the long computer time involved.

Our experiments have also revealed that the number of terms to be added increases with increasing complexity of the earth model. These seismograms are not shown here because alternate exact solutions are not available as well as because the number of terms considered by us in any of those cases was quite modest.

Finally, it is important to point out that Rao and Goda (1978) evaluated the integrals along a path of three segments (Figure 2, contour 1) in the k plane. The integration along their middle segment is analogous to our k integration except that their contour runs from P branch point to Rayleigh pole only while our contour goes much beyond the pole. On either side of Rao and Goda's middle segment there is a segment running oblique to the k_r axis until the integrand has diminished sufficiently in magnitude. This is an attractive strategy at first sight. But it is apparent that while the integrand diminishes monotonically along such oblique segments, the length of the path is infinite and the total contribution from the ignored sections may not be ignorable. What is important is to establish that a slight alteration of the path length along the contour, whether straight or segmented, will not alter the resulting seismogram radically. This is why we have been comparing seismograms for m and $m + \Delta m$.

CONCLUSION

On the basis of Figure 4 and numerous other computations we conclude that broad features of the synthetic seismograms may be obtained using numerical evaluation of the formal integral solutions of the boundary-value initial-value problems of seismic wave propagation. But accuracy in synthetic seismograms would be achieved by this method at considerable computational effort only.

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