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PARAMETRIC VIBRATION OF SHALLOW SPHERICAL SHELLS TRIANGULAR IN PLAN

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INTRODUCTION

Within the framework of linear theory, the basic equations of equilibrium of shallow spherical shells are well known in two types—(1) two coupled fourth degree partial differential equations in terms of a stress function and the normal deflection function and (2) three coupled partial differential equations in terms of three displacement functions. Thus the study of vibration for shallow spherical shells even with the neglect of tangential inertia forces demands for the solution of a complicated system of partial differential equations with a totality of eighth degree. It has, however, been proved by the author [1], without any extra similifying assumption, that the basic equations of equilibrium for shallow spherical shells can be reduced to the system below in cartesian co-ordinates-in terms of three displacements.

$$\left.\begin{aligned} \nabla u &= \lambda \frac{\partial w}{\partial x} \\ \nabla v &= \lambda \frac{\partial w}{\partial y} \\ \nabla \nabla w + \xi w &= q/D \end{aligned}\right\} \qquad \ldots(1)$$

where, ∇ = Laplacian Operator, $\lambda = (1 + \nu)/R$

R = radius of curvature of the shell,

ν = Poisson's ratio; E = Young's modulus,

h = shell thickness, D = flexural rigidity of the shell element, q = vertical load function and $\xi = Eh/DR^2$

It is obvious, that for investigating the transverse vibration of the shell ignoring the tangential inertia forces, the third equation of the system (1) is adequate by itself.

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EQUATION OF MOTION

In order to study the motion of the shell under parametric load $Z_0(x, y, t)$, the third equation of the system (1) will change to

$$\nabla\nabla w + \xi w = \frac{1}{D}\left(Z_0 + \Delta Z - m\frac{\partial^2 w}{\partial t^2}\right) \qquad ...(2)$$

where '*m*' is the mass of the shell/unit surface area, and ΔZ is the additional load arising due to the shell deformation under the action of Z_0 in a momentless state given by

$$\Delta Z = T_1 x_1 + T_2 x_2$$

where T_1, T_2 are the membrane forces
and

$$x_1 = -\frac{\partial^2 w}{\partial x^2},\ x_2 = -\frac{\partial^2 w}{\partial y^2}$$

It is assumed here that the vertical load Z_0 is uniform and equal to $-Q\cos\beta t$, so that membrane forces are

$$T_1 = T_2 = \frac{1}{2} RQ \cos\theta t$$

and the equation (2) becomes

$$\nabla\nabla w + \xi w + \frac{1}{D}\left(Q\cos\theta t + \frac{RQ}{2}\cos\theta t\ \nabla w + m\frac{\partial^2 w}{\partial t^2}\right) = 0 \qquad ...(3)$$

which, after taking a Laplace Operator, will transform to

$$\nabla\nabla\nabla w + \xi\nabla w + \frac{1}{D}\left(\frac{RQ}{2}\cos\theta t\ \nabla\nabla w + m\frac{\partial^2}{\partial t^2}\nabla w\right) = 0 \qquad ...(4)$$

Equation (4) represents the parametric vibration of shallow spherical shells. It is to be noted that the equation of motion for free vibration of the shell panels is directly obtained from eqn. (3) by assuming $Q = 0$ in the form :

$$\nabla\nabla w + \xi w + \frac{m}{D}\frac{\partial^2 w}{\partial t^2} = 0 \qquad ...(5)$$

If again, in the eqn. (4) representing the parametric vibration of the shell panels, it is assumed that $\theta = 0$, i.e. the normal load becomes static, and also the inertia term is ignored, the basic equation to determine the critical static load is obtained in the form:

$$\nabla\nabla\nabla w + \xi w + \frac{RQ}{2D}\nabla\nabla w = 0 \qquad ...(6)$$

ANALYSIS

3.1 Case 1: Simply-supported isosceles right-angled triangular shell panel of equal side lengths. The co-ordinate axes OX and OY (Fig. 1) are taken along the equal sides. The boundary conditions are given by

$$w = \partial^2 w/\partial x^2 = 0 \text{ for } x = 0;\ w = \partial^2 w/\partial y^2 = 0 \quad \text{for } y = 0$$

$$w = \frac{\partial^2 w}{\partial \mu^2} = 0 \text{ for } x + y = a;\ \frac{\partial}{\partial \mu} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$$

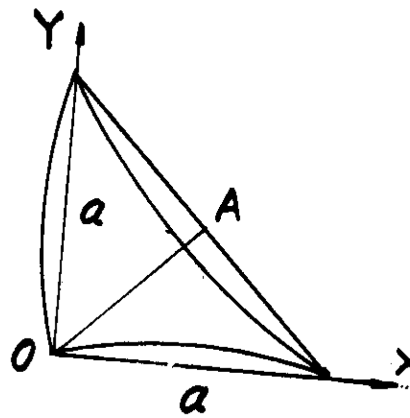

Fig. 1

Consistent with these boundary conditions, the displacement function $w\,(x, y, t)$ may be assumed in the form [3]

$$w\,(x, y, t) = f(t) \sum_{n=1,3,\ldots}^{\infty} A_n\, \phi_n\,(x, y) \qquad \ldots(7)$$

where

$$\phi_e\,(x, y) = \sin 2\beta x \cdot \sin \beta y + \sin \beta x \cdot \sin 2\beta y,\ \beta = \frac{n\pi}{a} \qquad \ldots(8)$$

and $f(t)$ is still an unknown function of time.

Substituting the expressions (7) and (8) into the equation (4), the following equation for $f(t)$ is obtained

$$\frac{d^2 f}{dt^2} + \frac{D}{m}\left(25\beta^4 + \xi - \frac{5RQ}{2D}\,\beta^2 \cos\,\theta t\right) \cdot f = 0 \qquad \ldots(9)$$

The frequencies of free vibrations 'p_n' of the shell panel are obtained by assuming the deflection function $w\,(x, y, t)$ in the form

$$w\,(x, y, t) = \sin pt \sum_{n=1,3\ldots}^{\infty} A_n\, \phi_n\,(x, y) \qquad \ldots(10)$$

and substituting it into the eqn. (5).
This gives:

$$p_n^2 = \frac{D}{m}\,(25\beta^4 + \xi) \qquad \ldots(11)$$

To determine the static critical load Q_n of the shell panel corresponding to the n-th mode of buckling, we assume the displacement function $w(\alpha, y)$ in the form

$$w(x, y) = \sum_{m=1,3,\ldots}^{\infty} A_n \phi_n(x, y) \qquad \ldots(12)$$

and substitute it in the eqn. (6). We obtain

$$Q_n^* = \frac{10D}{R}\beta^2 + \frac{2D\xi}{5R\beta^2} \qquad \ldots(13)$$

Now. the eqn. (9) for the parametric vibration can be written as:

$$\frac{d^2f}{dt^2} + \frac{D}{m}(25\beta^4 + \xi)\left\{1 - \frac{5RQ\beta^2 \cos\theta t}{2D(25\beta^4 + \xi)}\right\}\cdot f = 0 \qquad \ldots(14)$$

or

$$\frac{d^2f}{dt^2} + p_n^2(1 - 2\mu_n \cos\theta t)\cdot f = 0 \qquad \ldots(15)$$

where, 'p_n' is the frequencies for free vibrations as obtained in the eqn. (11) and,

$$2\mu_n = \frac{5RQ\beta^2}{2D(25\beta^4 + \xi)} = \frac{Q}{\frac{10D}{R}\beta^2 + \frac{2D\xi}{5R}\beta^2}$$

Thus, $2\mu_n = Q/Q^*$, by using eqn. (13).

3.2 Case 2: Simply-supported equilateral triangular panel (Figure 2).

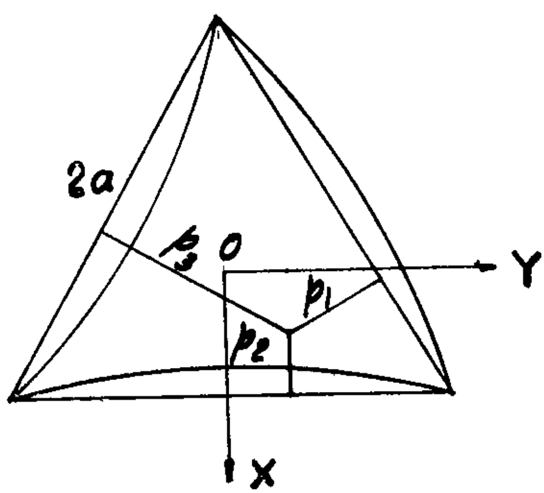

Fig. 2

Consider a simply-supported equilateral triangular shell panel with its centroid as the origin of co-ordinate axes. Using the trilinear co-ordinate system [2], the Laplacian Operator is transformed to

$$\nabla = \frac{\partial^2}{\partial p_1^2} + \frac{\partial^2}{\partial p_2^2} + \frac{\partial^2}{\partial p_3^2} - \frac{\partial^2}{\partial p_1 \partial p_2} - \frac{\partial^2}{\partial p_2 \partial p_3} - \frac{\partial^2}{\partial p_3 \partial p_1} \quad \text{...(16)}$$

where p_1, p_2 and p_3 are perpendiculars drawn from any point $p\ (x, y)$ inside the triangular region to the sides and are given by

$$p_1 = p_0 + \frac{x}{2} - \frac{\sqrt{3}}{2}\, y,\ p_2 = p_0 + \frac{x}{2} + \frac{\sqrt{3}}{2}\, y.\ p_3 = p_0 - x$$

such that $p_1 + p_2 + p_3 = \text{constant} = 3p_0 = K$ (say).

Here p_0 is the radius of the inscribed circle.

The boundary conditions in the new co-ordinate system are

$$w = \nabla w = 0, \text{ for } p_1 = p_2 = p_3 = 0 \quad \text{...(17)}$$

For the solution of the equation (4) consistent with boundary conditions (17) it is assumed that

$$w\,(x,\ y,\ t) = f(t) \sum_{n=1,2,\dots}^{\infty} B_n \psi_n\,(p_1, p_2, p_3) \quad \text{...(18)}$$

where,

$$\psi_n\,(p_1, p_2, p_3) = \sin \eta p_1 + \sin \eta p_2 + \sin \eta p_3,\ \eta = \frac{2n\pi}{K} \quad \text{...(19)}$$

Substituting the expression (18) into the equation (4), the equation to determine $f(t)$ is obtained in the form

$$\frac{d^2 f}{dt^2} + \frac{D}{m}\left(\eta^4 + \xi - \frac{RQ}{2D}\,\eta^2 \cos \theta t\right)\cdot f = 0 \quad \text{...(20)}$$

For frequencies of free vibration 'p_n' of the shell panel, we assume the deflection function $w\,(x, y, t)$ in the form:

$$w\,(x,\ y,\ t) = \sin pt \sum_{n=1,2,\dots}^{\infty} B_n \psi_n\,(p_1, p_2, p_3) \quad \text{...(21)}$$

and substitute it into the eqn. (5). We get:

$$p_n^2 = \frac{D}{m}\,(\eta^4 + \xi) \quad \text{...(22)}$$

The static critical load Q_n of the shell panel corresponding to the n-th mode of buckling is obtained by assuming the displacement function $w\,(x, y)$ in the form:

$$w\,(x,\ y) = \sum_{1,2,\dots}^{\infty} B_n \psi_n\,(p_1, p_2, p_3) \quad \text{...(23)}$$

and substituting it into the eqn. (6). Thus it is obtained,

$$Q_n^* = \frac{2D}{R}\eta^2 + \frac{2D\xi}{R\eta^2} \qquad ...(24)$$

Now, the eqn. (20) for the parametric vibration can be written in the form:

$$\frac{d^2f}{dt^2} + \frac{D}{m}(\eta^4+\xi)\left\{1 - \frac{RQ\eta^2 \cos\theta t}{2D(\eta^4+\xi)}\right\}\cdot f - 0 \qquad ...(25)$$

or

$$\frac{d^2f}{dt^2} + p_n^2(1 - 2\mu_n \cos\theta t)\cdot f = 0 \qquad ...(26)$$

where, 'p_n' is spectra of frequencies of free vibration given by the eqn. (22), and

$$2\mu_n = \frac{RQ\eta^2}{2D(\eta^4+\xi)} = \frac{Q}{\frac{2D}{R}\eta^2 + \frac{2D\xi}{R\eta^2}}$$

Thus, $2\mu_n = Q/Q_n^*$, by using eqn. (24)

DISCUSSION

It is seen that the equations for parametric excitation in both the cases are transformed to the well-known Mathieu equation

$$\frac{d^2f}{dt^2} + p_n^2(1 - 2\mu_n \cos\theta t)\cdot f(t) = 0 \qquad ...(27)$$

where 'p_n' denotes the frequency spectra for free vibration, and is given by

$$p_n^2 = \frac{D}{m}(25\beta^4 + \xi), \quad \text{in the first case}$$

and

$$p_n^2 = \frac{D}{m}(\eta^4 + \xi), \quad \text{in the second case} \qquad ...(28)$$

and 'μ_n' is known as the coefficient of excitation and equal to

$$\mu_n = \frac{Q}{2Q_n^*} \qquad ...(29)$$

where Q_n^* denotes the critical load under statical condition, and is given by

$$Q_n^* = \frac{10D}{R}\beta^2 + \frac{2D\xi}{5R\beta^2}, \quad \text{in the first case}$$

and

$$Q_n^* = \frac{2D}{R}\eta^2 + \frac{2D\xi}{R\eta^2}, \quad \text{in the second case} \qquad ...(30)$$

It is to be observed that $\xi = 0$ corresponds to the plate problem having same shape and boundary conditions.

If p_n^0 is the spectre of frequencies for free vibrations of the corresponding plates, then

$$\left.\begin{aligned} p_n^0 &= 5\beta^2\sqrt{D/m}, && \text{for the first case} \\ \text{and} \quad p_n^0 &= \eta^2\sqrt{D/m}. && \text{for the second case} \end{aligned}\right\} \quad \ldots(31)$$

Expressions (28) may be rewritten by using eqn. (31) in the form

$$p_n = p_n^0\sqrt{1 + d} \quad \ldots(32)$$

$$\left.\begin{aligned} \text{where,} \quad d &= \xi/25\beta^4, && \text{for the first case} \\ \text{and,} \quad d &= \zeta/\eta^4, && \text{for the second case} \end{aligned}\right\} \quad \ldots(33)$$

Substituting the values for ξ, η, β ie the expressions (33), and introducing a non-dimensional parameter γ, such that

$$\gamma = \frac{12\,(1 - \nu^2)\,a^4}{n^4h^2R^2} \quad \ldots(34)$$

it can be obtained that

$$\left.\begin{aligned} d &= \gamma/25n^4, && \text{for the first case} \\ \text{and,} \quad d &= 9\gamma/16n^4, && \text{for the second case} \end{aligned}\right\} \quad \ldots(35)$$

It is obvious that the factor 'd' gives the necessary correction for shell action over the corresponding thin plate.

The first three regions of dynamic instability of the problem defined by equation (27) are approximately given by [4]

$$\left.\begin{aligned} \theta^* &= 2p_n\sqrt{1 \pm \mu_n} \\ \theta^* &= p_n\sqrt{1 + \mu_n^2/3} \\ \theta^* &= p_n\sqrt{1 - 2\mu_n^2} \\ \theta^* &= \frac{2}{3}\,p_n\sqrt{1 - \frac{9\mu_n^2}{8 \pm 9\mu_n}} \end{aligned}\right\} \quad \ldots(36)$$

Table 1-2 showing the regions of instability for different values of the parameter 'γ' are provided with μ_n varying from 0.0 to 0.4 in terms of θ^*/p_n^0 with $n = 1$ for the two cases. A table-3 is also provided showing the ratio p_n/p_n^0 for different values of γ with $n = 1$ for both the cases.

TABLE 1. Zones of Instability for case I

	μ_1	1st Zone		2nd Zone		3rd Zone	
		1	2	1	2	1	2
$\gamma \neq 0$	0.0	2.0000	2.0000	1.0000	1.0000	0.6667	0.6667
	0.1	2.0976	1.8974	1·0017	0.9899	0.6633	0.6624
	0.2	2.1909	1.7889	1.0066	0.9592	0.6543	0.6470
	0.3	2.2804	1.6733	1.0149	0.9055	0·6409	0.6136
$\gamma = 60$	0.0	3.6878	3.6878	1.8439	1.8439	1.2293	1.2293
	0.1	3.3678	3.4936	1.8470	1.8254	1.2230	1.2215
	0.2	4.0398	3·2985	1.8562	1.7686	1.2065	1.1931
	0.3	4.2048	3.0854	1.8714	1.6697	1.1818	1 1314
$\gamma = 120$	0.0	4.8166	4.8166	2 4083	2.4083	1.6055	1.6055
	0·1	5.0517	4.5695	2.4123	2.3841	1.5974	1.5953
	0.2	5.2764	4.3081	2.4243	2.3100	1.5758	1.5582
	0.3	5.4918	4.0299	2·4442	2.1808	1·5436	1.4778
$\gamma = 180$	0.0	5.7271	5.7271	2.8636	2.8636	1.9090	1.9090
	0.1	6.0067	5.4332	2.8683	2.8346	1.8994	1.8969
	0.2	6.2738	5.1225	2.8826	2.7466	1.8773	1.8528
	0.3	6.5299	4.7917	2.9062	2.5931	1.8354	1.7571

TABLE 2. Zones of Instability for case II

	μ_1	1st Zone		2nd Zone		3rd Zone	
		1	2	1	2	1	2
$\gamma \neq 60$	0.0	11.7898	11.7898	5.8949	5.8949	3.9299	3.9299
	0.1	12.3653	11.1848	5.9047	5.8357	3.9100	5.9050
	0.2	12.9151	10.5461	5.9341	5.6542	3.8571	3.8141
	0.3	13.4425	9.8651	5.9827	5.3381	3.7783	3.6172
	0.4	13.9499	9.1324	6.0501	4.8611	3.6779	3.2233
$\gamma = 120$	0.0	16.5529	16.5529	8.2765	8.2765	5.5176	5.5176
	0.1	17.3609	15.7035	8.2903	8.1933	5.4897	5.4826
	0.2	18.1328	14.8054	8.3315	7.9385	5.4154	5.3551
	0.3	18.8733	13.8492	8.3997	7.4947	5.3047	5.0785
	0.4	19.5857	12.8219	8.4943	6.8250	5.1638	4.5256
$\gamma = 180$	0.0	20.2237	20.2237	10.1119	10.1119	6.7412	6.7412
	0.1	21.2108	19.1859	10.1287	10.0102	6.7071	6.6984
	0.2	22.1540	18.0887	10.1791	9.6990	6.6163	6.5426
	0.3	23.0586	16.9204	10.2624	9.1567	6.4811	6.2048
	0.4	23.9291	15.6652	10.3780	8.3385	6.3090	5.5292

TABLE-3: Value of p_n/p^0_n for different values of γ for $n = 1$.

p_n/p_n^0 \ γ	20	40	60	80	100	140	180
Case 1.	1.3415	1.6125	1.8439	2.0494	2.2361	2.5690	2.8636
Case 2.	3.5000	4.8477	5.8949	6.7823	7.5664	8.9303	10.1118

REFERENCES

1. B. Bhattacharyya 1978 Procs. Institution of Civil Engineers 65 (2), 421-429. Analysis of Shallow Spherical Shells in the form of an equilateral triangle in plan.
2. B. Sen 1968 Bulletin of the Calcutta Mathematical Society 60, 31-35. Trilinear co-ordinates and boundary value problems.
3. B. Bhattacharyya 1977 Journal of Sound and vibration 54 (3), 464-467. Free vibration of plates on Vlasov's foundation.
4. V. V. Bolotin 1964, Dynamic stability of elastic system, V. I. Weingarten et al, Holden-Day Inc., Sanfransisco.