Paper No. 157, Bulletin, I. S. E. T. Vol. 12, No. 3, pp. 120–127 September 1975

# DYNAMIC ANALYSIS OF PLANE FRAMES WITH SHEAR WALLS

B. K. GOYAL,* S.P. SHARMA* AND S.K. AGARWAL*

## INTRODUCTION

With the increase in heights of multistoreyed buildings and revision of seismic coefficients in the country modal analysis of tall buildings is desirable. This paper presents an efficient method for the dynamic analysis of frames considering effects of bending, axial and shear deformations. The distributed mass of vertical members is replaced by lumps at beam levels. Other assumptions remain the same as in the case of a static analysis. (Goyal and Sharma, 1968)

## FORMULATION

Generation of stiffness matrix for plane frames with shear walls has been discussed elsewhere (Goyal and Sharma, 1968) and is briefly summarised below.

Figure 1 shows a typical plane frame with shear wall. To explain the assembly technique the derivation of the stiffness matrix for the first column only is presented, others can be treated in a similar manner.

Let $K_{11}$ and $K_{12}$ be the stiffness matrices of a member connecting joints 1 and 2 in an absolute coordinate system. Using the standard assembly technique, equilibrium of joints 1 to $m$ can be written as shown below. Lateral loads acting in $X$-direction only are considered.

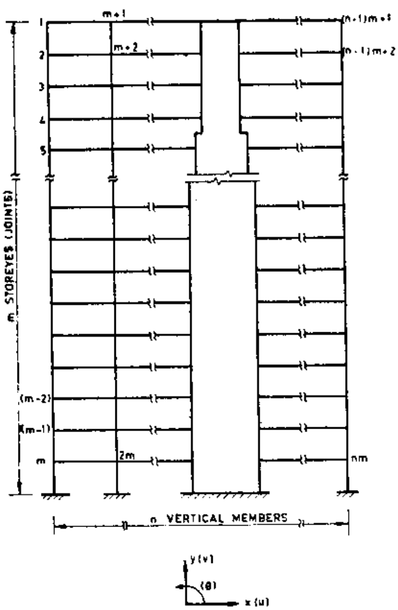


Fig. 1. Joint Numbering and Coordinate System of a Frame

*Structural Engineering Research Centre, Roorkee (U.P.) India.

$$\begin{bmatrix} \Sigma[k'_{11}]_{3\times3} & [k'_{12}]_{3\times3} & \cdot & \cdot & \cdot & \cdot & [k'_{1\ m+1}]_{3\times3} & & & \\ [k'_{21}] & \Sigma[k'_{22}] & [k'_{23}] & \cdot & \cdot & \cdot & \cdot & [k'_{2\ m+2}] & & \\ & [k'_{32}] & \Sigma[k'_{33}] & [k'_{34}] & \cdot & \cdot & \cdot & \cdot & [k'_{3\ m+3}] & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & [k'_{m\ m-1}] & \Sigma[k'_{mm}] & \cdot & \cdot & \cdot & & [k'_{m\ 2m}] \end{bmatrix}_{(3m\times6m)}$$

$$\times \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \\ d_{m+1} \\ \vdots \\ d_{2m} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \\ f_{m+1} \\ \vdots \\ f_{2m} \end{bmatrix}$$

where,

$$d_r=\{u_r\ \ v_r\ \ \theta_r\}$$

and

$$f_r=\{p_r\ \ 0\ \ 0\}$$

are the deformations and the force vectors for a joint $r$. This matrix may be written in the following form:

$$k_{11}\delta_1+k_{12}\delta_2=F_1$$

where

$$k_{11}=\begin{bmatrix} \Sigma[k'_{11}] & [k'_{12}] & & & \\ [k'_{21}] & \Sigma[k'_{22}] & [k'_{23}] & & \\ & [k'_{32}] & \Sigma[k'_{33}] & [k'_{34}] & \\ & \cdot & \cdot & \cdot & \cdot \\ & & & [k'_{m\ m-1}] & \Sigma[k'_{mm}] \end{bmatrix}$$

$$k_{12}=\begin{bmatrix} [k'_{1\ m+1}] & & & & \\ & [k'_{2\ m+2}] & & & \\ & & [k'_{3\ m+3}] & & \\ & & & \cdot & \\ & & & & [k'_{m\ 2m}] \end{bmatrix}$$

$$\delta_1=\{d_1\ \ d_2\ \ .\ .\ .\ \ d_m\}$$
$$\delta_2=\{d_{m+1}\ d_{m+2}\ \ .\ .\ \ d_{2m}\}$$

and

$$F_1=\{f_1\ f_2\ \ .\ .\ .\ \ f_m\}$$

It would be noticed that $\delta_r$ is the deformation of the $r$th column/shear wall and $F_r$ the loads acting on it. It is assumed that $F_1$ to $F_{n-1}$ are zero—all lateral loads act only on the last vertical member and is denoted by $F$.

$$F=\{p_{r+1}\ \ 0\ \ 0\ \ p_{r+2}\ \ 0\ \ 0\ \ .\ .\ .\ p_{r+m}\ \ 0\ \ 0\},$$

where

$$r=(n-1)\,m$$

The conditions of equilibrium for the entire frame may now be expressed as shown below

$$k_{11}\delta_1+k_{12}\delta_2=0$$
$$k_{21}\delta_1+k_{22}\delta_2+k_{23}\delta_3=0$$
$$k_{32}\delta_2+k_{33}\delta_3+k_{34}\delta_4=0 \qquad (1)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$k_{n\,n-1}\,\delta_{n-1}+k_{nn}\,\delta_n=F$$

From Equation (1*a*)

$$\delta_1=-k_{11}^{-1}\,k_{12}\,\delta_2$$

Substituting $\delta_1$ in Equation (1b)

$$[k_{22}-k_{21}\,k_{11}^{-1}\,k_{12}]\;\delta_2+k_{23}\,\delta_3=0$$

or

$$\bar{k}_{22}\,\delta_2+\bar{k}_{23}\,\delta_3=0$$

where

$$\bar{k}_{22}=k_{22}-k_{21}\,k_{11}^{-1}\,k_{12}$$

Proceeding similarly

$$\bar{k}_{33}\,\delta_3+\bar{k}_{34}\delta_4=0$$

where

$$\bar{k}_{33}=k_{33}-k_{32}\,\bar{k}_{22}^{-1}\,k_{23}$$

and finally

$$\bar{k}_{n-1\,n-1}\,\delta_{n-1}+k_{n-1\,n}\,\delta_n=0$$

where

$$\bar{k}_{n-1\,n-1}=k_{n-1\,n-1}-k_{n-1\,n-2}\,\bar{k}_{n-2\,n-2}^{-1}\,k_{n-2\,n-1}$$

or

$$\delta_{n-1}=-\bar{k}_{n-1\,n-1}^{-1}\,k_{n-1\,n}\,\delta_n$$

Substituting for $\delta_{n-1}$ in Equation (1)

$$[k_{nn}-k_{n\,n-1}-\bar{k}_{n-1\,n-1}^{-1}\,k_{n-1\,n}]\,\delta_n=F$$

or

$$\bar{k}_{nn}\,\delta_n=F$$

Since $F$ has non zero component in $x$ direction only $k_{nn}$, $\delta_n$ and $F$ can be rearranged in the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} U \\ \phi \end{bmatrix}\;\begin{bmatrix} P \\ O \end{bmatrix} \qquad (2)$$

where

$$U=\{u_{r+1} \quad u_{r+2}\ldots u_{r+m}\}$$
$$\phi=\{v_{r+1} \quad \theta_{r+1} \quad v_{r+2} \quad \theta_{r+2}\ldots v_{r+m} \quad \theta_{r+m}\}$$
$$P=\{p_{r+1} \quad p_{r+2}\ldots p_{r+m}\};\; r=(n-1)m$$

Partitioning the matrices of Equation (2) and eliminating for $\phi$ it can be shown that

$$\phi=-A_{22}^{-1}\,A_{21}\,U \qquad (3)$$

$$AU=P \qquad (4)$$

where

$$A=A_{11}-A_{12}\,A_{22}^{-1}\,A_{21}$$

[*A*] is thus the stiffness matrix of the frame for horizontal deformations $u$. For the purposes of dynamic analysis the masses are lumped at storey levels reducing the number of unknowns to $m$ instead of the number of joints in the frame $nm$. The mode shapes and natural frequencies can now be found by computing eigen values and eigen vectors of the matrix

$$[M^{-\frac{1}{2}} A \, M^{-\frac{1}{2}}]$$

where $[M]$ is the appropriate mass matrix. Horizontal forces are then calculated giving the vector $\{P\}$. The horizontal deformations $\{U\}$ of the $n$th vertical member are obtained using Equation (4). The other two deformations for this vertical member $\{\phi\}$ are available from Equation (3).

Rest of the deformations may now be obtained by back substitution. Bending moment, axial force and shear force at member ends can then be calculated.

## COMPUTING TECHNIQUE

Tridiagonalisation is used to obtain the condensed stiffness matrix of the frame. This procedure reduces the storage required substantially. Stiffness matrix for one vertical member need only be stored ($K_{ii}$, $i=1\ldots m$, size $3m\times 3m$). Matrix for beams $K_{ij}$ ($i=1, \ldots n, j=1, \ldots m$) is a diagonal matrix of $3\times 3$ submatrices—one corresponding to each beam. Thus only the submatrices need be stored.

## SEQUENCE OF COMPUTATION

Matrix $K_{12}$ can be written in the form

$$K_{12}=\begin{bmatrix} [B_1]_{(3\times3)} & & & & \\ & [B_2]_{(3\times3)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & [B_n] \end{bmatrix}_{(3n\times 3n)}$$

Let $K_{11}^{-1}$ be defined as given below

$$K_{11}^{-1}=\begin{bmatrix} k_{11} & k_{12} & \cdot & \cdot & \cdot & k_{1n} \\ k_{21} & k_{22} & \cdot & \cdot & \cdot & k_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_{n1} & k_{n2} & \cdot & \cdot & \cdot & k_{nn} \end{bmatrix}$$

Hence

$$K_{11}^{-1} K_{12}=\begin{bmatrix} k_{11} B_1 & k_{12} B_2 & \cdot & \cdot & k_{1n} B_n \\ k_{21} B_1 & k_{22} B_2 & \cdot & \cdot & k_{2n} B_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ k_{n1} B_1 & k_{n2} B_2 & \cdot & \cdot & k_{nn} B_n \end{bmatrix}$$

where

$$[B_r] \equiv [k'_{r\,m+r}]$$

Since this matrix is required for back substitution it is punched out or stored on tape/disc for a subsequent use. Further,

$$K_{21}=K_{12}^T=\begin{bmatrix} B_1^T & & & & \\ & B_2^T & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & B_n^T \end{bmatrix}$$

It can be easily shown that

$$K_{21}K_{11}^{-1}K_{12}=\begin{bmatrix} B_1^T\,k_{11}\,B_1 & B_1^T\,k_{12}\,B_2 & \cdot\;\cdot\;\cdot & B_1^T\,k_{1n}\,B_n \\ B_2^T\,k_{21}\,B_1 & 1_2^T\,k_{22}\,B_2 & \cdot\;\cdot\;\cdot & B_2^T\,k_{2n}\,B_n \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ B_n^T\,k_{n1}B_1 & B_n^T\,k_{n2}\,B_2 & \cdot\;\cdot\;\cdot & B_n^T\,k_{nn}\,B_n \end{bmatrix} \quad (5)$$

Equation (5) thus gives an algorithm for calculating the matrix product required for the generation of matrix [A] used in equation (4).

Intermediate output is used in back substitution.

## ILLUSTRATIVE EXAMPLES

Two examples have been solved using this approach. Experimental results were available for the first in the literature. The second one is chosen to demonstrate the utility of the method for a frame likely to occur in practice. Eigen values were computed using the Jacobi Classical Transformation Technique.

## EXAMPLE I

Tso and Chang (1971) have conducted an experimental investigation on coupled shear wall model made out of plexiglas. Value of the fundamental frequency with only inertia forces acting has been reported. They, however, have not commented on the probable experimental errors. Figure 2 shows the dimensions of the model. Effect of local wall

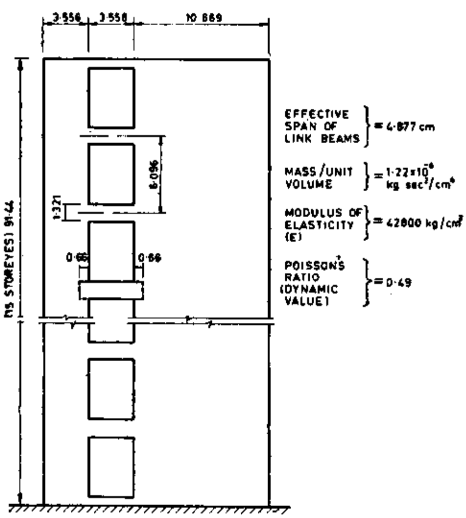


Fig. 2. Details of Frame Solved in Example 1

deformations has been considered as recommended by Michael (1966). The predicted value, using the proposed approach, is found to be 63.29 c.p.s. while the reported experimental value is 58.4 c.p.s.

Tso and Chang report that their predicted value was 5% above the experimental value. Value obtained by the proposed simplified approach is about 8½% higher than the experimental value. It would be seen that the simplifying assumption which resulted in a drastic reduction in storage requirement and computation time has given reasonably accurate results.

The difference may also partly be attributed to the replacement of distributed mass by discrete masses. This effect would be prominent because of the absence of floor loads. It is expected that in an actual structure carrying sizeable floor loads the error would be smaller.

First three mode shapes and corresponding frequencies are shown in Fig. 3.

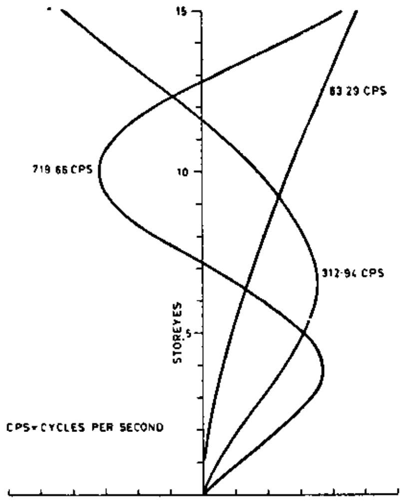


Fig. 3. Mode Shapes and Frequencies for Example 1

## EXAMPLE 2

Figure 4 shows a typical frame having a shear wall and columns—a configuration likely to occur in practice. First three mode shapes and corresponding frequencies are given in Fig. 5.

## CONCLUDING REMARKS

The proposed approach permits calculation of mode shapes and frequencies of plane frames with shear walls and subsequent determination of member forces with a relatively small computational effort. Calculations can be carried out even on medium sized computers. Accuracy of prediction is expected to be sufficient for design purposes.

The Fortran programme developed can handle 18 storeyed frames (number of bays is not limited) on an IBM 1620 (60K) machine. Frames with shear walls of variable cross section can also be analysed.

## ACKNOWLEDGEMENT

This work was carried out at the Structural Engineering Research Centre, Roorkee (U.P.) India.

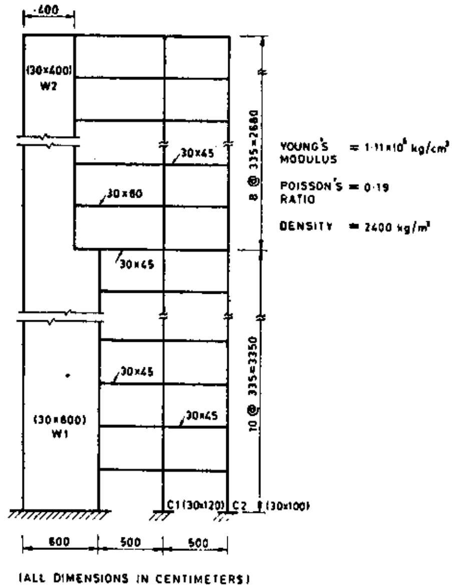


**Fig. 4. Details of Frame solved in Example 2.**

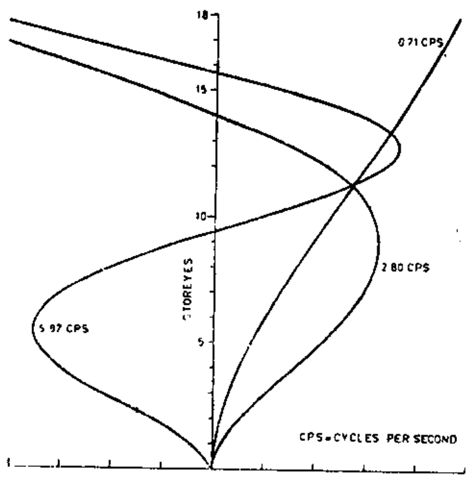


Fig. 5. Mode Shapes and Frequencies for Example 2

## REFERENCES

1. Goyal, B.K. and Sharma, S.P.—"Matrix Analysis of Frames with Shear Walls," Building Science Vol. 3, 1968, pp. 93–98.
2. Michael, D.—"The effect of local wall deformations on the elastic interaction of cross walls coupled by beams" Proceedings of the Symposium on tall buildings, Southampton, April 1966, pp. 253–270.
3. Tso, W.K. and Chang, H.—"Dynamic Analysis of Plane Coupled Shear Walls". Journal of Engineering Mechanics Division, Proceeding of American Society of Civil Engineers, Vol. 97, No. EM 1, 1971, pp. 33–48.

## APPENDIX—NOTATION

$x$, $y$—Cartesian Coordinate system

$\theta$—Rotation about normal to $x-y$ plane

$u$, $v$—Translation along the $x$ and y directions respectively

$d_r = \{u_r \quad v_r \quad \theta_r\}$
the joint deformation vector of $r$

$\delta_i = \{d_{r+1}\ d_{r+2} \ldots d_{r+m}\}$, $r = (i-1)m$
the deformation vector of a vertical member $i$

$p_i$—Load in the $x$ direction at joint $i$

$F$—Total load vector acting at the last vertical member

$[k'_{ik}]$—Stiffness matrix of the member $ik$

$[M]$—Mass matrix

$[B_r] \equiv [K'_{r\ m+r}]$