THE VIBRATIONS OF A CLAMPED RECTANGULAR PLATE WITH CONCENTRATED MASS, SPRING AND DASHPOT

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SYNOPSIS

An approximate analytical method has been given for the determination of natural frequencies of a composite system consisting of an isotropic rectangular plate with a concentrated mass, spring and dashpot attached at any point of the plate, the plate being clamped at all the edges. This method makes use of a double series expansion in terms of the beam function. Numerical examples are given for a square plate with (a) concentrated mass at the centre and (b) a spring at the centre. This method is applicable to many other edge conditions and combinations of mass, spring and dashpot.

Nomenclature

2a, 2b	sides of the rectangular plate
h	plate thickness
D	flexural rigidity of the plate
E	Young's modulus of the plate material
υ	Poisson's ratio
K_p	spring constant of the plate
K_s	constant of the plate per unit area
ρ	mass of the plate per unit area
M_p	total mass of the plate
M	concentrated mass
c	dashpot strength
γ	b/a ratio
μ	exponential decay constant
p	circular frequency of the system
W(x,y,t)	deflection of the plate
X_m, Y_n	beam functions
λ	the frequency parameter

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1. Introduction.

In engineering instrumentation or in structural analysis, problems often arise where the natural frequencies of lateral vibrations of a rectangular plate of appreciable mass which carries a concentrated mass are important, especially when the attached masses are comparable in magnitude to the mass of the plate itself. Under such circumstances the natural frequencies of the plate-mass system may be considerably different from those of the plate without masses. For a vibrating beam with concentrated mass, spring and dashpot an analytical solution has been given by Dana Young (1948). This method makes use of a series expansion in terms of the set of orthogonal functions which represent the normal modes of vibration of the beam alone. This method is very general in character and may be applied when the beam has any type of end supports. This method has been extended by Das and Navaratna (1963) to isotropic rectangular plates with attached mass, spring and dashpot. They have considered a rectangular plate simply supported along two parallel edges and supported in any manner along the other two parallel edges. A Fourier series expansion in terms of the corresponding plate-eigenfunctions has been utilised to represent the modal form of the plate system. An independant analysis for the problem of vibration of a plate with attached mass has been given by Thein Wah (1961). Even here the plate is simply supported on two opposite edges. An extension of Young's procedure is possible only for plates treated by the above authors where the two opposite edges are simply supported. Only in this case a series expansion in terms of the plate eigenfunctions is possible. For other types of boundary conditions such an expansion is not possible as the plate eigenfunctions are not known. This paper is devoted to an analytical solution for such plates.

A fourier series procedure has been made use of in solving the vibrations of a clamped rectangular plate with concentrated mass, spring and dashpot. It leads to the same results as given by the Galerkin method when the same functions are used. The procedure may be applied for any combination of clamped and simply supported edges. When there are free edges it is not possible to use this method and the Rayleigh-Ritz method may have to be used.

2. The equation for the composite system.

Let the mass, spring and dashpot be attached to the plate at a point (x_1, y_1) (Fig. 1.). During vibration the plate may be considered to be under forced vibration due to the force of interaction between the plate and the mass-spring-dashpot system. Then the motion of the plate is described by

$$D \nabla^4 W + \rho \frac{\partial^2 W}{\partial t^2} = f(x, y)\, e^{(-\mu + ip)\, t} \qquad (1)$$

Where μ is the exponential decay constant and $f(x, y)\, e^{(-\mu+ip)\, t}$ represents the force of interaction.

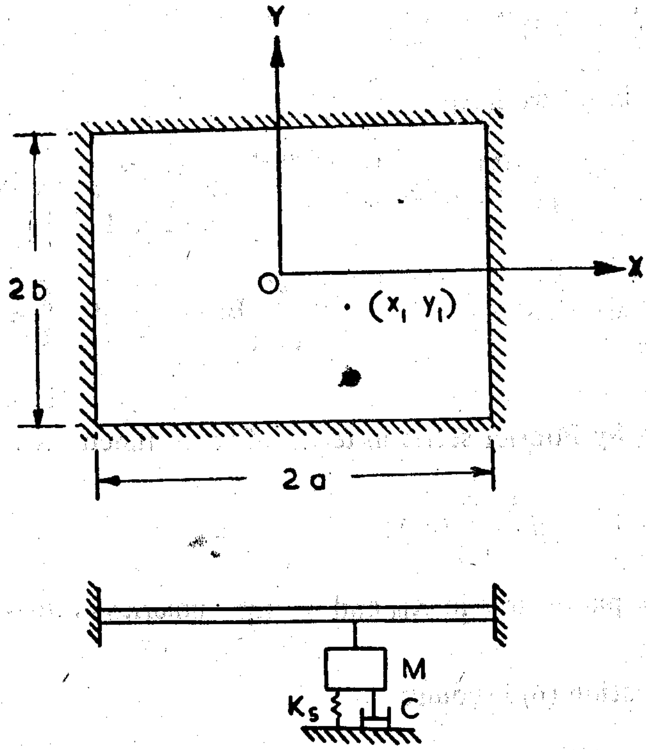

Figure 1

The differential equation for the mass-spring-dashpot system can be written as

$$M \frac{d^2 \overline{W}}{d t^2} + c \frac{d \overline{W}}{d t} + K_s \overline{W} = -F_o e^{(-\mu + ip) t} \quad (2)$$

Where $\overline{W} = W (x_1, y_1, t)$

$f(x, y) = \delta (x_1, y_1) F_o$; $\delta (x_1, y_1)$ being the Dirac-delta function in two dimensions, c and K_s are the dashpot strength and the spring constant respectively.

Combining (1) and (2) we may write

$$\nabla^4 W + \frac{\rho}{D} \frac{\partial^2 W}{\partial t^2} + \frac{\delta(x_1, y_1)}{D} \left[M \ddot{\overline{W}} + c \dot{\overline{W}} + K_s \overline{W} \right] = 0 \quad (3)$$

The solution may now be assumed in the form

$$W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} X_m Y_n e^{(-\mu + ip) t} \quad (4)$$

where X_m, Y_n are the eigenfunctions of a clamped beam. We now expand $\delta (x_1, y_1)$ by a fourier series—

$$\delta (x_1, y_1) = \sum_{m-1}^{\infty} \sum_{n-1}^{\infty} A_{mn} X_m Y_n \quad (5)$$

where $A_{mn} = \frac{1}{4ab} X_m(x_1)\, Y_n(y_1)$

Substituting (5) and (4) in (3) we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\left\{\alpha^4_m+\beta^4_n+\frac{e(ip-\mu)^2}{D}\right\}X_mY_ne^{(-\mu+ip)t}+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}X''_mY''_n e^{(-\mu+ip)t}$$

$$+\frac{1}{4Dab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_m(x_1)Y_n(y_1)X_mY_n e^{(-\mu+ip)t}\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}X_i(x_1)Y_j(y_1)\left\{M(ip-\mu)^2+c(ip-\mu)+K_s\right\}\right] = 0 \quad (6)$$

Expanding X''_m and Y''_n by Fourier series in terms of beam functions we can write

$$X''_m = \alpha^2_m \sum_{i=1}^{\infty} K^m_i X_i\ ;\ Y''_n = \beta^2_n \sum_{j=1}^{\infty} L^n_j Y_j \quad (7)$$

where α_m and β_n are the parameters in X_m and Y_n the numerical values of which will be given later.

Using (7) the equation (6) becomes

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[B_{mn}\left\{\alpha^4_m+\beta^4_n+\frac{\rho\,(ip-\mu)^2}{D}\right\}+2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}\,\alpha^2_i\,\beta^2_j\,K^i_mL^j_n\right]X_m\,Y_n\,e^{(-\mu+ip)t}$$

$$+\frac{1}{4\,Dab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_m(x_1)Y_n(y_1)X_mY_n e^{(-\mu+ip)t}\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}X_i(x_1)Y_j(y_1)\left\{M(ip-\mu)^2+c(ip-\mu)\right.\right.$$

$$\left.\left.+K_s\right\}\right]=0 \quad (8)$$

Putting

$$C^{mn}_{mn}=a^2b^2\,(\alpha^4_m+\beta^4_n+2\alpha^2_m\beta^2_n\,K^m_m\,L^n_n\,)$$

$$C^{mn}_{ij}=2\,a^2\,b^2\,\alpha_i^2\,\beta_j^2\,K^i_mL^j_n,\quad i\neq m,\ \text{or}\ j\neq n$$

$$E^{mn}_{ij} = X_m\,(x_1)\,Y_n\,(y_1)\,X_i\,(x_1)\,Y_j\,(y_1)$$

and collecting the coefficient of each $X_m\, Y_n e^{(-\mu+ip)\,t}$ in (8) and equating to zero we get

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}\left[C^{mn}_{ij}+\frac{\rho(\mu^2-p^2)a^2b^2}{D}\delta^{mn}_{ij}+\frac{E^{mn}_{ij}ab}{4D}\left\{M\,(\mu^2-p^2)+K_s-c\mu\right\}\right]=0 \quad (9)$$

$m = 1, 2, 3\ldots\ldots\ldots\ldots$

$n = 1, 2, 3\ldots\ldots\ldots\ldots$

and

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}\left[\frac{2\mu\, p\rho\, a^2 b^2}{D}\delta_{ij}^{mn} + \frac{E_{ij}^{mn} ab}{4D}(2\mu\, pM - cp)\right] = 0 \qquad (10)$$

m = 1, 2, 3...........
n = 1, 2, 3...........

Thus we get two infinite sets of homogeneous equations in the unknowns B_{ij}. The non-dimensional parameters involving μ and p can be found from the condition that the two infinite determinants of coefficients shall vanish for non-trivial solutions.

3. Plate with a single concentrated mass M.

We get this case by putting $K_s = c = \mu = 0$ in the above equations. The set of equations (10) vanishes identically and the set (9) reduces to

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} F_{ij}\left[C_{ij}^{mn} - \lambda\left(\delta_{ij}^{mn} + \frac{M}{M_p}E_{ij}^{mn}\right)\right] = 0 \qquad (11)$$

m = 1, 2, 3...........
n = 1, 2, 3...........

Numerical work has been carried out for the case of a square clamped plate with a concentrated mass at centre. Considering only symmetric vibrations, the beam functions selected are

$$X_m = \frac{\cosh \alpha_m x}{\cosh \alpha_m a} - \frac{\cos \alpha_m x}{\cos \alpha_m a}$$

$$Y_n = \frac{\cosh \beta_n y}{\cosh \beta_n b} - \frac{\cos \beta_n y}{\cos \beta_n b} \qquad (12)$$

The values of $\alpha_m a$ and $\beta_n b$ are given in the Table-1 , below.

TABLE 1

m	$\alpha_m a = \beta_m b$
1	2.3650204
2	5.4978039
3	8.6393798
m>3	$(4m-1)\frac{\pi}{4}$

Taking third order determinats, approximations to the first three symmetric modes have been given. The convergence has been studied by allowing the first order determinant, the second order determinant and the third order determinant to vanish successively. These calculations have been carried out for several values of M/M_p ratio and the results have been given in the figures 2 and 3. It was found that the convergence of the values for λ for the first mode is good. To obtain better values one will have to consider higher order deter-

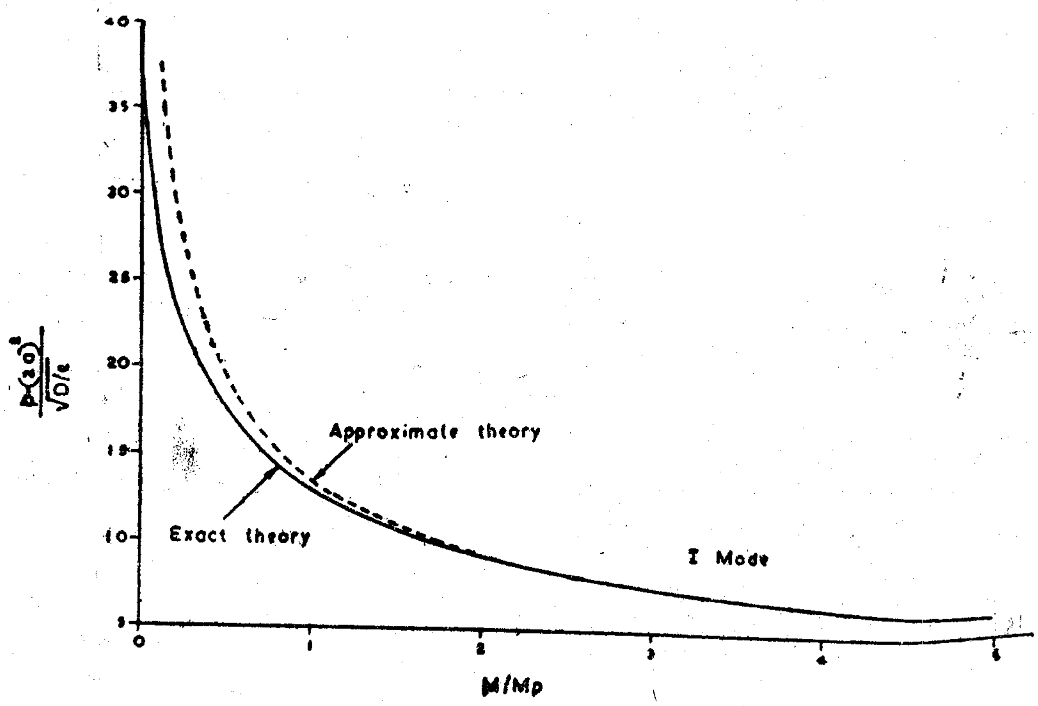

Figure 2

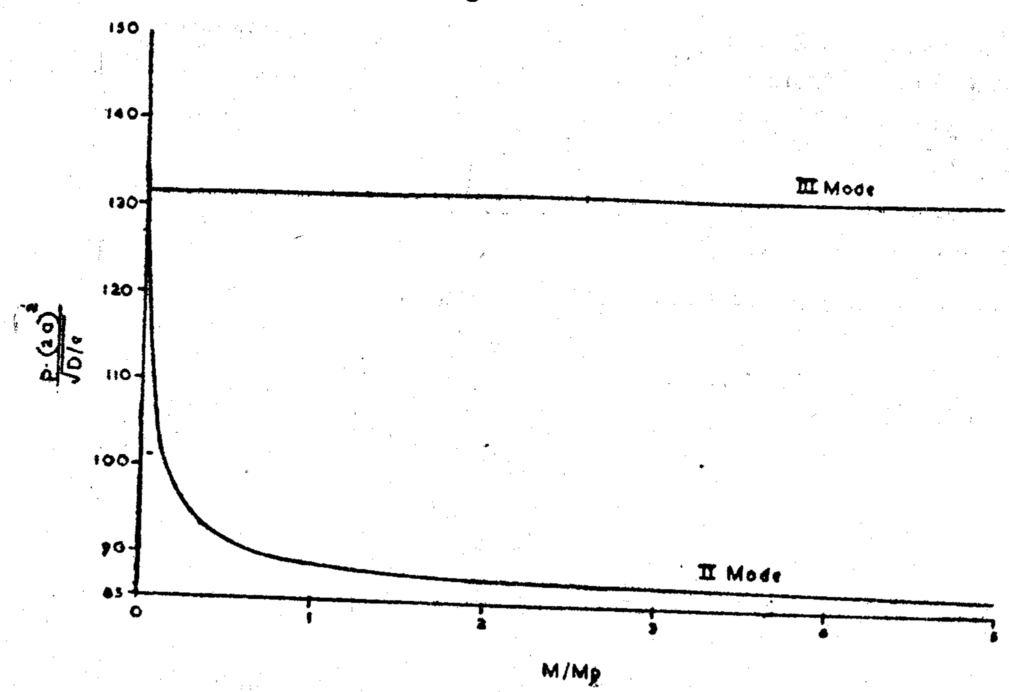

Figure 3

minants. It was also found that for $M/M_p = 0.0022$ values of λ for the second and third modes are equal. This happens because of the presence of a nodal point at the centre in the second mode of an ordinary plate without concentrated mass. The presence of mass leaves the frequency of this mode unaffected while that of the third mode is reduced. Hence for values of $M/M_p > 0{\cdot}0022$ the order of the modes get interchanged.

When the M/M_p ratio is very large it is customary to approximate the system by a single degree of freedom system for the first mode of vibration. In the approximation the plate is replaced by an equivalent spring having a spring constant K_p. This spring constant (Timoshenko Krieger 1959) K_p has a Value of $D/0{\cdot}0224\ a^2$ for a square plate of side 2a. Using this we may write down the square of the circular frequency of the system as

$$p^2 = K_p/M = D/0{\cdot}0224\ Ma^2$$

This may be rewritten as

$$\frac{p.\,(2a)^2}{\sqrt{D/\rho}} = \frac{13{\cdot}4}{\sqrt{M/M_p}}$$

The variation of this frequency parameter has been presented in Fig. 2. in dotted lines. It may be noticed that for values of M/M_p greater than unity this approximate theory differs from the exact theory by less than 5 percent. The two frequencies become indistinguishable for M/M_p greater than 2.25.

4. Plate with a spring.

The solution for this case is obtained by putting c, μ and M equal to zero in the set of equations (9) and (10). The set (10) vanishes identically and the set (9) reduces to

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}\left[C_{ij}^{mn} - \frac{\rho p^2a^2b^2}{D}\delta_{ij}^{mn} + \frac{E_{ij}^{mn}ab}{4D}K_s\right] = 0 \qquad (13)$$

$$m = 1, 2, 3 \ldots\ldots\ldots\ldots$$
$$n = 1, 2, 3 \ldots\ldots\ldots\ldots$$

Again putting $K_p = \dfrac{D}{0{\cdot}0224\, a^2}$ this set can be written as

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}\left[C_{ij}^{mn} - \frac{\rho\, p^2a^2b^2}{D}\delta_{ij}^{mn} + E_{ij}^{mn}\frac{b}{a}\cdot\frac{K_s}{K_p}\cdot\frac{1}{0{\cdot}0896}\right] = 0 \qquad (14)$$

$$m = 1, 2, 3 \ldots\ldots\ldots\ldots$$
$$n = 1, 2, 3 \ldots\ldots\ldots\ldots$$

Here again we obtain an infinite determinant and the approximation to λ are determined as described in Art. 3. The numerical calculations have been carried out for a square plate with a spring at the centre, considering symmetric vibrations just as in Art. 3. The results are given in Fig. 4. It may be noticed that the centre of the plate happens to be a nodal point for the second mode.

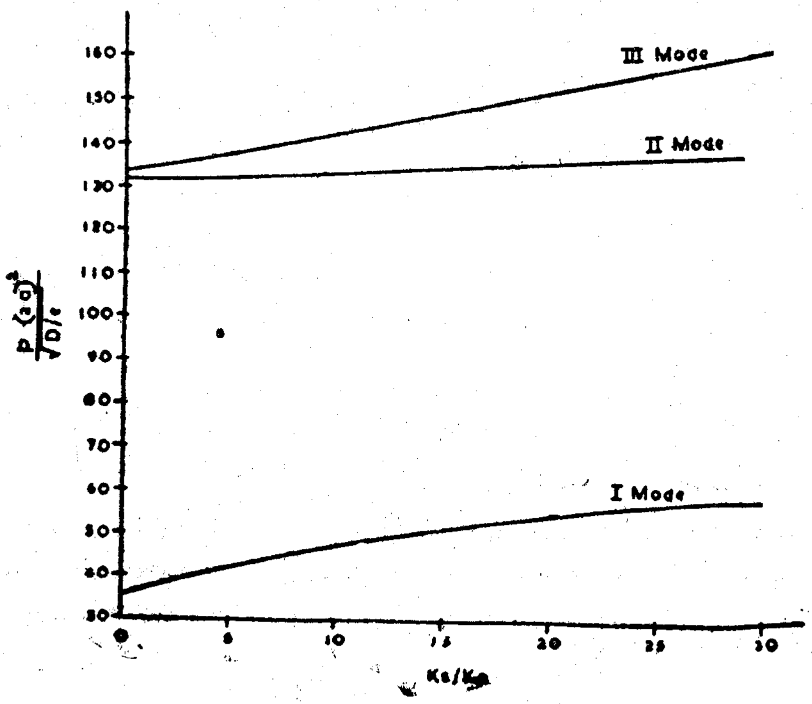

Figure 4

5. Plate with a dashpot

We obtain this case by putting M and K_s equal to zero. The following non-dimensional parameters have been introduced for convenience.

$$\lambda_1 = \frac{\rho(\mu^2-p^2)\, a^2b^2}{D}, \quad X = \frac{c\,\mu\, ab}{4D}$$

$$\overline{\lambda_1} = \frac{2\mu p\, \rho a^2b^2}{D}, \quad \overline{X} = \frac{c\, p\, a\, b}{4D} \tag{15}$$

We now obtain the infinite sets of homogeneous equations from (9) and (10)

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}\left[C_{ij}^{mn} + \lambda_1\, \delta_{ij}^{mn} - X\, E_{ij}^{mn}\right] = 0 \tag{16}$$

m = 1, 2, 3............

n = 1, 2, 3............

and

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}\left[\overline{\lambda_1}\, \delta_{ij}^{mn} - \overline{X}\, E_{ij}^{mn}\right] = 0 \tag{17}$$

m = 1, 2, 3............

n = 1, 2, 3............

From the relations (15) we have

$$\frac{\rho\,(\mu+ip)^2\,a^2\,b^2}{D} = \lambda_1 + i\,\bar{\lambda}_1 \tag{18}$$

and

$$(\mu + ip)^2 = \frac{16\,D^2}{c^2 a^2\,b^2}\,(X + i\,\bar{X})^2 \tag{19}$$

Using (19) in (18) we get

$$\lambda_1 = \frac{16\,\rho\,D}{c^2}\,(X^2 - \bar{X}^2) \tag{20}$$

$$\bar{\lambda}_1 = \frac{32\,\rho\,D}{c^2}\,X\bar{X} \tag{21}$$

The values of $\bar{\lambda}_1$, λ_1, X and $\bar{X}$ must now be determined such that the determinants corresponding to (16) and (17) vanish and the relations (20) and (21) are satisfied. The numerical labour in such a determination is quite involved and would require the use of a digital computer.

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