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BEAM VIBRATIONS BY METHOD OF INITIAL PARAMETERS

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Summary

Method of initial parameter, originally developed for static problems is extended to solve the initial value and forced vibration problems of beams. Two examples are worked out. This method is particularly useful in cases of concentrated forces and moments, and time dependent boundry conditions.

Introduction

Analysis of beam vibrations due to initial conditions as well as due to external exciting forces is a classical one. Vibration problem can be analysed both by method of separation of variables and oprational methods, where natural frequencies are obtained through a characteristic equation, and then the corresponding mode shapes are found. Using the orthoganality property, that exists, among mode shapes, initial value broblem as well as forced vibration problem can be solved.

In this paper, method of initial parameters, originally developed for static problems (1, 2)** is extended to beam vibration problems. Applying Laplace transform to Euler-Bernoulli equation, the equation governing the beam vibration is reduced to contain only the space variable. This transformed equation along with the transformed initial and boundary conditions is sol ved by the method of initial parameters. The inverse transform then gives the complete solution.

Method of Analysis

The Euler-Bernoulli equation governing the transverse vibration of a uniform beam is

$$EI \frac{\partial^4 w}{\partial x^4}(x, t) + m \frac{\partial^2 w}{\partial t^2}(x, t) = q(x, t). \tag{1}$$

Where

E I	=	flexural rigidity of a beam
m	=	mass per unit length
w (x, t)	=	transverse deflection
q (x, t)	=	is the external exciting force

The initial condition are

$$w(x, o) = y_o(x) \ ; \ \left.\frac{\partial w}{\partial t}\right|_{t=o} = \dot{y}_o(x). \tag{2}$$

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** Numbers in brackets designate references at end of paper.

The beam has four boundry conditions, two at each end. There may be any combination of hinged, clamped, free and elastic supports. Taking the Laplace transform of equation (1), we obtain

$$\frac{d^4 \overline{W}}{d x^4}(x, s) + \frac{m}{E I} s^2 \overline{W}(x, s) = \frac{\overline{q}}{E I}(x, s) + \frac{m}{E I} s\, y_0(x) \frac{m}{E I} \dot{y}_0(x) \tag{3}$$

where $\overline{g}(x, s) = \int_0^{\infty} g(x, t)\, e^{-st}\, dt$ and s is a transformed parameter.

The solution of the homogeneous part of this equation (3) is

$$\overline{W}(x, s) = A \sin \beta x + B \cos \beta x + C \sinh \beta x + D \cosh \beta x. \tag{4}$$

where

$$\beta^4 = -\frac{m}{E I} s^2 . \tag{5}$$

Define the initial parameters as follows :

$$\begin{aligned} \overline{W}(o, s) &= \overline{W}_0(s) \\ \overline{\theta}(o, s &= \overline{W}'(o,s) = \overline{\theta}_0(s) \\ \overline{M}(o, s) &= -E I \overline{W}''(o, s) = \overline{M}_0(s) \\ \overline{V}(o, s) &= -E I \overline{W}'''(o, s) = \overline{V}_0(s), \end{aligned} \tag{6}$$

where prime indicates derivative with respect to x. Using the equation (6) the four constants A, B, C and D in equation (4) can be found in terms of initial parameters $\overline{W}_0$, $\overline{\theta}_0$, $\overline{M}_0$ and $\overline{V}_0$. The equation (4) in terms of initial parameters will be

$$\overline{W}(x, s) = \overline{W}_0 f_1(\beta x) + \frac{\overline{\theta}_0}{\beta} f_2(\beta x) + \frac{\overline{M}_0}{EI \beta^2} f_3(\beta x) + \frac{\overline{V}_0}{E I \beta^3} f_4(\beta x), \tag{7}$$

where

$$\begin{aligned} f_1(\beta x) &= \tfrac{1}{2}(\cos \beta x + \cosh \beta x) \\ f_2(\beta x) &= \tfrac{1}{2}(\sin \beta x + \sinh \beta x) \\ f_3(\beta x) &= \tfrac{1}{2}(\cos \beta x - \cosh \beta x) \\ f_4(\beta x) &= \tfrac{1}{2}(\sin \beta x - \sinh \beta x). \end{aligned} \tag{8}$$

Case 1 : Beam having a concentrated load P (t) at $x = \xi$. (Figure 1). Since we know two of the four initial parameters at $x = 0$, there will remain only two unknown parameters in equation (7). For the range $0 \leq x < \xi$ the equation (7) holds good as there is no transverse load on the beam. We can write

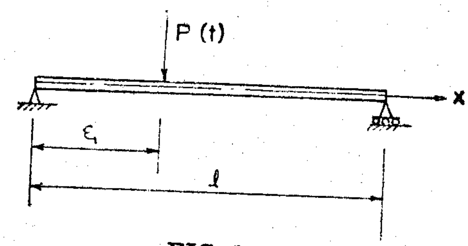

FIG. 1

$$\overline{W}_1(x, s) = \overline{W}(x, s) \qquad 0 \leq x < \xi \tag{9}$$

Since we have a concentrated load at $x=\xi$ we can consider this as a jump in shear and write down the shear condition as

$$\overline{v}(\xi, s) = -\overline{P}(s). \tag{10}$$

For the range $\xi < x \leq l$, $\overline{W}(x, s) = \overline{W}_1(x, s) + \overline{W}_2(\bar{x}, s)$ (11)

where $w_2(\bar{x}, s)$ represents the additional deflection due to the jump in shear given in equation (10), and $\bar{x} = x - \xi$.

To find $\overline{W}_2(\bar{x}, s)$, let us imagine a fictitious beam for the range $\xi < x \leq l$.

For this beam the following are the initial parameters :

$$\begin{aligned}
&\overline{W}_2\,(o,\,s) \;=0\\
&\overline{\theta_2}\;(o,\,s)\;=0\\
&\overline{M}_2\,(o,\,s) = -\,E\,I\,\overline{W}_2''\,(o,\,s) = 0\\
&\overline{V}_2\,(o,\,s) = -\,E\,I\,\overline{W}_2''' = -\,\overline{P}\,(s).
\end{aligned} \tag{12}$$

$\overline{W}_2\,(\bar{x},\,s)$ is in the form of equation (7) except instead of x we have $\bar{x}$. Using equation (12), we obtain

$$\overline{W}_2\,(\bar{x},\,s) = -\,\frac{\overline{P}\,(s)}{E\,I\,\beta^3}\,f_4\,(\beta\,\bar{x}). \tag{13}$$

Hence, the complete solution, equation (11), can be written as

$$\overline{W}\,(x,\,s) = w_o\,f_1\,(\beta x) + \frac{\overline{\theta_o}}{\beta}\,f_2\,(\beta x) + \frac{\overline{M}_o}{E\,I\,\beta^2}\,f_3\,(\beta x) + \frac{\overline{V}_o}{E\,I\,\beta^3}\,f_4\,(\beta x) - \frac{\overline{P}\,(s)}{E\,I\,\beta^3}\,f_4\,(\beta\,\bar{x}). \tag{14}$$

The remaining two parameters can be found from the transformed boundary conditions at x=l. Taking the inverse Laplace transform of equation (14), we can obtain the required solution w (x, t).

Case 2 : Beam having a concentrated moment M_c (t) at x=ξ. (Figure 2). In this case there is a jump in moment at x=ξ. Following in a similar manners as Case 1 solution can be obtained as

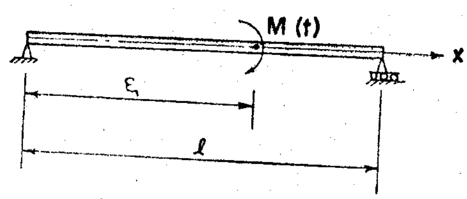

FIG. 2

$$\overline{W}\,(x,\,s) = \overline{W}_o\,f_1\,(\beta x) + \frac{\overline{\theta_o}}{\beta}\,f_2\,(\beta x) + \frac{\overline{M}_o}{E\,I\,\beta^2}\,f_3\,(\beta x) + \frac{\overline{V}_o}{E\,I\,\beta^3}\,f_4\,(\beta x) - \frac{\overline{M}_c\,(s)}{E\,I\,\beta^2}\,f_3\,(\beta\,\bar{x}). \tag{15}$$

Case 3 : Beam subjected to arbitary load of intensity Q (x, t). (Figure 3).

In this case we can take

$\overline{P}\,(s) = \overline{Q}\,(\xi,\,s)\,d\,s$ at $x=\xi$ and $l_1 \leq \xi \leq l_2$.

Following in a similar manner as Case 1, we can write

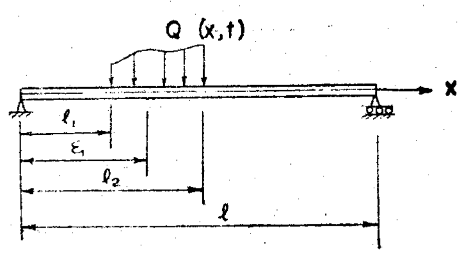

FIG. 3

$$\overline{W}\,(x,\,s) = \overline{W}_o\,f_1\,(\beta x) + \frac{\overline{\theta_o}}{\beta}\,f_2\,(\beta x) + \frac{\overline{M}_o}{E\,I\,\beta^2}\,f_3\,(\beta x) + \frac{\overline{V}_o}{E\,I\,\beta^3}\,f_4\,(\beta\,x) - $$

$$-\,\frac{1}{E\,I\,\beta^3}\int_{l_1}^{x}\overline{Q}\,(\xi,\,s)\,f_4\,(\beta\,\bar{x})\,d\,\xi. \tag{16}$$

Considering $\overline{Q}\,(\xi,s) = \frac{\overline{q}}{EI}(\xi,\,s) + \frac{m\,s}{E\,I}\;y_o\,(\xi) + \frac{m}{E\,I}\,\dot{y}_o\,(\xi)$, equation (16) represents solution to initial value problem as well as forced vibration problem.

Example 1 : Forced-vibration of a simply supported beam with a time dependent load at an arbitary point on the beam.

In this case the initial displacement and initial velocity of the beam are taken as zero [i.e., $y_o\,(x) = \dot{y}_o(x) = 0$].

The solution for this problem is given by equation (14) with the following values of the initial parameters.

$$\bar{W}_0 = 0$$
$$\bar{M}_0 = 0 \tag{17}$$
$$\bar{\theta}_0 = \frac{\bar{P}(s)}{EI\beta^3}\left\{\frac{f_4(\beta \bar{l})\, f_2(\beta l) - f_2(\beta \bar{l})\, f_4(\beta l)}{f_2^2(\beta l) - f_4^2(\beta l)}\right\}$$
$$\bar{V}_0 = \bar{P}(s)\left\{\frac{f_2(\beta l)\, f_2(\beta \bar{l}) - f_4(\beta l)\, f_3(\beta \bar{l})}{f_2^2(\beta l) - f_4^2(\beta l)}\right\}$$

Substituting equation (17) into equation (14), the transformed deflection is given by

$$\bar{W}(x,s) = \frac{\bar{P}(s)}{2EI} \cdot \frac{1}{\beta^3}\left\{\frac{\sin\beta\xi \sin\beta(l-x)}{\sin\beta l} - \frac{\sinh\beta\xi \sinh\beta(l-x)}{\sinh\beta l}\right\} \tag{18}$$

After taking inverse Laplace transform of equation (18), the deflection is given by

$$w(x,t) = \int_0^t P(t-\tau)\, F(\tau)\, d\tau \tag{19}$$

where

$$F(t) = \sum_{n=1}^{\infty} \frac{2}{m\, l\, \omega_n} \sin\frac{n\pi\xi}{l} \cdot \sin\frac{n\pi x}{l} \cdot \sin\omega_n t \tag{20}$$

and

$$\omega_n = n^2\pi^2\sqrt{\frac{EI}{m\, l^4}} \tag{21}$$

Taking $P(t) = A_0 \sin\Omega t$, the deflection is given by

$$w(x,t) = \frac{2A_0}{m\, l}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi\xi}{l} \cdot \sin\frac{n\pi x}{l}}{\omega_n} \frac{\omega_n \sin\Omega t - \Omega\sin\omega_n t}{(\omega_n^2 - \Omega^2)} \tag{22}$$

Equation (22) checks with the value given in Reference [3].

Example 2 : A simply supported beam subjected to prescribed end motion.

Here the boundry conditions are

$$w(o,t) = 0$$
$$w''(o,i) = 0$$
$$w(l,t) = f(t)$$
$$w''(l,t) = 0$$

The solution is still given by equation (14) without the last term, with the following values of the initial parameters

$$\bar{W}_0 = 0$$
$$\bar{M}_0 = 0$$
$$\bar{\theta}_0 = \bar{f}(s) \cdot \frac{\beta f_2(\beta l)}{f_2^2(\beta l) - f_4^2(\beta l)} \tag{23}$$
$$\bar{V}_0 = -\bar{f}(s) \cdot EI\beta^3 \frac{f_4^2(\beta l)}{f_2^2(\beta l) - f_4^2(\beta l)} \cdot$$

The transformed deflection is given by

$$\bar{w}(x, s) = \bar{f}(s) \frac{\sin \beta x \sinh \beta l + \sin h\beta x \sin \beta l}{2 \sin \beta l \sinh \beta l} \tag{24}$$

After the inverse transform, the deflection is given by

$$w(x, t) = \int_0^t f(t-\tau)\, F(\tau)\, d\tau \tag{26}$$

where

$$F(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\omega_n}{n\pi} \cdot \sin \frac{n \pi x}{l} \sin \omega_n t \tag{26}$$

where

$$\omega_n = n^2 \pi^2 \sqrt{\frac{E I}{m l^4}}$$

Taking $f(t) = B_0 \sin \Omega t$, the deflection is given by

$$w(x, t) = B_0 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2 \omega_n}{n \pi (\omega_n^2) - \Omega^2)} (\omega_n \sin \Omega t - \Omega \sin \omega_n t) \sin \frac{n \pi x}{l} \tag{27}$$

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