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DISPERSION OF LOVE-WAVES IN A TRANSVERSELY ISOTROPIC INHOMOGENEOUS LAYER LYING OVER A HOMOGENEOUS HALF-SPACE

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INTRODUCTION

Pec (1967) used the ray theory to derive the frequency equation of Love–waves propagating in a vertically inhomogeneous, isotropic medium. Bhattacharya (1970) studied the propagation of Love-waves in an inhomogeneous, isotropic layer over a homogeneous half-space. The shear-wave velocity and the modulus of rigidity in the layer were assumed to vary linearly along the direction of propagation and it was shown that the phase velocity of love-waves decreases or increases according to their direction of propagation along the increasing or decreasing shear-wave velocity and the rigidity of the layer. Negi and Singh (1973) extended the investigations of Bhattacharya by considering the inhomogeneities of exponential type in lateral and vertical directions. The variations of velocity and rigidity were assumed to be of the type

$$\beta_1(x) = \beta_0 \exp(ax) \quad \text{and} \quad \mu_1 = \mu_0 \exp(bx) \text{ respectively.}$$

Vlaar (1968) developed ray theory for an inhomogeneous, anisotropic medium, based on the concept of a wave front as the carrier of discontinuity in particle velocity and derived the equations for rays and travel times for the case of transversely-isotropic, vertically inhomogeneous medium. Negi and Singh (1972) applied this theory for an inhomogeneous, anisotropic medium, in which the variations of rigidities and the density in the layer are of such a type that the horizontal shear-wave velocity remains constant whereas the vertical shear-wave velocity varies linearly with depth.

In the present case, we consider the propagation of Love-waves in an inhomogeneous, transversely isotropic layer of thickness H lying over a homogeneous, isotropic half-space. The vertical and horizontal rigidities and the density in the layer are assumed to vary as

$$\mu_V = \mu_{V_0} \exp(p+m)z, \quad \mu_H = \mu_{H_0} \exp(pz) \quad \text{and} \quad \rho(z) = \rho_0 \exp(pz),$$

where p and m are certain constants. The ray theory technique is used

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to get the frequency equation. This has been solved for two particular models.

FORMULATION OF THE PROBLEM

We consider a medium consisting of an inhomogeneous, transversely isotropic layer of thickness H lying over a homogeneous, isotropic half-space. The upper surface of the layer is supposed to be stress free. We take the origin in the free surface and x-axis in the direction of the propagation of the wave. Z-axis is taken vertically downwards into the medium. Let the incident ray AB make an angle θ_0 at the point B with the x-axis as indicated in Figure 1.

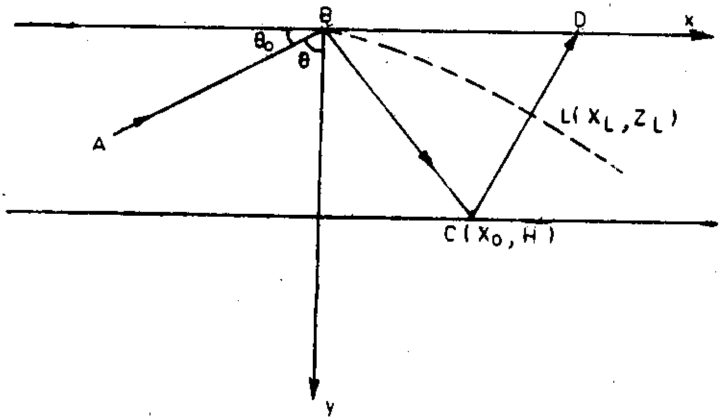

Fig. 1 Diagram Showsng the Model and Incident Ray

The variations of elastic parameters in the layer are assumed to be

$$\mu_V = \mu_{V_0} \exp(p+m)z$$
$$\mu_H = \mu_{H_0} \exp(pz) \quad (1)$$
$$\rho(z) = \rho_0 \exp(pz)$$

where μ_{V_0}, μ_{H_0}, ρ_0 are the corresponding values at the free surface and p and m are certain constants.

The velocity and rigidity in the half-space are denoted by β_2 and μ_2 respectively. The differential equations for down going and up going rays (BC and CD) in such a medium are respectively, given by (Vlaar, 1968)

$$\frac{dx}{dz} = \pm \frac{NP}{[L(\rho - P^2N)]^{1/2}} \quad (2)$$

where N and L are directiona rigidities and P is the ray parameter.

Let $\theta = (90° - \theta_0)$ be the angle of incidence of the ray AB. The ray parameter P is given by (Sato and Lapwood, 1968)

$$P = \frac{V_R \cdot \cos\theta}{\beta_{H_0}^2}, \tag{3}$$

where V_R is the velocity of energy-propagation in the layer and β_{H_0} is the horizontal shear wave velocity. The expression for the energy transmission in the transversely-isotropic, vertically inhomogeneous medium is given by

$$V_R = \left(\frac{\sin^2\theta}{\beta_H^2} + \frac{\cos^2\theta}{\beta_V^2}\right)^{-1/2} \tag{4}$$

where β_H and β_V are the horizontal and vertical shear-wave velocities in the medium and θ is the angle of incidence of the ray at the boundary. Using (4) in (3), the ray parameter P at the boundary is given by

$$P = \frac{\beta_{V_0} \sin\theta_0}{\beta_{H_0}[\beta_{H_0}^2 \cdot \sin^2\theta_0 + \beta_{V_0}^2 \cdot \cos^2\theta_0]^{1/2}} \tag{5}$$

Now the condition of constructive interference is

$$\phi_{BCL} - \delta = 2n\pi, \quad \text{where } n = 0, 1, 2 \tag{6}$$

where δ is the phase change due to the reflection at the point C and ϕ_{BCL} is the phase change from B to Z_L.

The z-coordinate of the point of intersection of up-coming rays and its orthogonal trajectories passing through the point B, will be determined so that the phase change can easily be found by integrating the equation of phase change. Making the substitutions from (1) in (2), the differential equation of down-going ray BC is

$$\frac{dx}{dz} = \frac{e^{mz}}{\xi[a^2 - e^{mz}]^{1/2}}$$

where $$a = \frac{1}{P\beta_{V_0}} \quad \text{and} \quad \xi = \rho\left[\frac{\mu_{H_0}}{\mu_{V_0}}\right] \tag{7}$$

Integrating the above equation, we obtain

$$x = \frac{1}{\xi}\int_0^z \frac{\exp(mz)}{\sqrt{(a^2 - \exp(mz))}}\,dz$$

$$= \frac{2}{m\xi}\left(a - \sqrt{(a^2 - \exp(mz))}\right) \tag{8}$$

The differential equation of the up-coming ray, CD, which is reflected at the interface is

$$\frac{dx}{dz} = -\frac{\exp(mz)}{\xi\sqrt{(a^2 - \exp(mz))}} \tag{9}$$

Thus the equation of the up-coming ray is obtained by integrating the above equation from the point (X_C, H) to any (x, z) in the layer (Figure 1) and is given by

$$x = x_C + \int_H^z \frac{\exp(mz)}{\sqrt{(a^2 - \exp(mz))}}\, dz$$

$$= x_C + \frac{2}{m\xi}[\sqrt{(a^2 - \exp(mz))} - \sqrt{(a^2 - \exp(mH))}] \quad (10)$$

Since, in such a medium, the direction of propagation of energy of *SH*-body waves and its wave front are perpendicular to each other, the differential equation of the wave front for the up-coming ray passing through B will be, from (9)

$$\frac{dx}{dz} = \xi \sqrt{(a^2 - \exp(mz))}/\exp(mz)$$

Integration of the above equation between the limits 0 to z leads to

$$x = \frac{2\xi}{m}\Big[-\sqrt{(a^2 - \exp(mz))}/2 \cdot \exp(mz) + \frac{1}{2a}\log\Big[\frac{a + \sqrt{(a^2 - \exp(mz))}}{\exp(mz/2)}\Big]$$

$$+ \sqrt{(a^2 - 1)}/2 - \frac{1}{2a}\log[a + \sqrt{(a^2 - 1)}]\Big] \quad (11)$$

Let $L(X_L, Z_L)$ be the point of intersection of the up-coming ray CD and the wave front of the same ray passing through the origin. To obtain the co-ordinates of the point L, we equate (10) and (11) and use the relation (8) to get

$$\frac{1}{\xi^2}(a^2 - \exp(mZ_L))^{\frac{1}{2}} + \frac{(a^2 - \exp(mZ_L))^{\frac{1}{2}}}{2 \cdot \exp(mZ_L)} - \frac{1}{2a}\log\Big[\frac{a + (a^2 - \exp(mZ_L))^{\frac{1}{2}}}{\exp(mZ_L)/2}\Big]$$

$$= \frac{2}{\xi^2}(a^2 - e^{mH})^{1/2} + \frac{\sqrt{(a^2 - 1)}}{2} - \frac{1}{2a} \cdot \log[a + (a^2 - 1)^{1/2}] - \frac{a}{\xi^2} \quad (12)$$

The numerical values of dimensionless parameter mz_L $(= \bar{Z}_L)$ for different angles of incidence and different numerical values of elastic parameters are obtained and are given in Tables 1 and 2.

If $k_1(z) = \frac{\omega}{V_R(z)}$ be the wave number of *SH*-body waves in the layer, the phase change, $d\phi$, along the path of the ray for an infinitesimal distance 'ds' is given by

$$d\phi = k_1(z)\, ds = \frac{\omega}{V_R(z)}\sqrt{\Big[1 + \Big(\frac{dx}{dz}\Big)^2\Big]} \cdot dz \quad (13)$$

Now $\quad \beta_H = \frac{\mu_H(z)}{\rho(z)}$ and $\beta_V = \frac{\mu_V(z)}{\rho(z)}$. Therefore,

V_R from (4) is given by

$$V_R = \frac{\beta_{H_0}\cdot\exp\left(\frac{mz}{2}\right)}{[\cos^2\theta_0\cdot\exp(mz)+\eta^2\cdot\sin^2\theta_0]^{1/2}} \tag{14}$$

where $\eta = \frac{\beta_{H_0}}{\beta_{V_0}}$

Therefore, from (13) and (14), we have

$$d\phi = \frac{\omega}{\beta_{H_0}\cdot\exp\left(\frac{mz}{2}\right)}[\cos^2\theta_0\cdot\exp(mz)+\eta^2\sin^2\theta_0]^{1/2}\cdot\sqrt{\left[1+\left(\frac{dx}{dz}\right)^2\right]}\,dz$$

The total phase change along the path BCL is given by

$$\phi_{BCL} = \frac{\omega}{\beta_{H_0}}\int_0^z[\exp(mz)\cdot\cos^2\theta_0+\eta^2\sin^2\theta_0]^{1/2}\cdot\left[1+\frac{\exp(2mz)}{\xi^2(a^2-e^{mz})}\right]^{1/2}\frac{1}{e^{mz/2}}\,dz$$

$$= \frac{\omega}{m\beta_{H_0}}\int_1^{\exp(\bar{z}_L)}[x\cos^2\theta_0+\eta^2\sin^2\theta_0]^{1/2}\cdot\left[1+\frac{x^2}{\xi^2(a^2-x)}\right]^{1/2}\cdot\frac{1}{x^{3/2}}\,dx$$

$$= \frac{K_0 I}{m}, \tag{15}$$

where $K_0 = \frac{\omega}{\beta_{H_0}}$

and

$$I = \int_1^{\exp(\bar{z}_L)}[x\cos^2\theta_0+\eta^2\sin^2\theta_0]^{1/2}\cdot\left[1+\frac{x^2}{\xi^2(a^2-x)}\right]^{1/2}\frac{1}{x^{3/2}}\,dx \tag{15a}$$

The values of I are calculated approximately with the help of Simpson's rule for different values of θ_0 and are given in Tables 1 and 2.

Let $k_0 = \frac{\omega}{\beta_{H_0}} = K\psi$, in which $\psi = \frac{c}{\beta_{H_0}} = \frac{1}{\cos\theta_0}$ and $k = \frac{\omega}{c}$,

c being the phase velocity of Love-waves in the medium. The values of ψ lie between 1 and β_2/β_{H_0}.

For numerical evaluation, two different models are considered. The values of numerical constants are taken from the works of Negi and Singh (1972). In the first case, the numerical values of the dimensionless parameter Z_L $(= mz)$ and I, for different values of ψ and $mH = 0.28$, $\mu_{H_0} = 3.50\times10^{10}$ dynes/cm^2,

$\mu_{V_0} = 3.10\times10^{10}$ dynes/cm^2, $\beta_{H_0} = 3.57\times10^5$ cm/sec.

$\beta_{V_0} = 3.36\times10^5$ cm/sec., $\xi = 1.06$, $\beta_{H_0}/\beta_2 = .70$ are obtained and given in Table 1. For the second case the corresponding values are given in Table 2, when $mH = .28$, $\mu_{V_0} = 3.32 \times 10^{10}$ dynes/cm^2, $\mu_{H_0} = 3.20 \times 10^{10}$ dynes/cm^2, $\beta_{H_0} = 3.42 \times 10^5$ cm/sec. $\beta_{V_0} = 3.48 \times 10^5$ cm/sec., $\xi = .98$, $\beta_{H_0}/\beta_2 = .80$ and $\beta_2 = 6.5 \times 10^{10}$ dynes/cm^2.

Now the phase change (δ) due to the reflection of the ray at the point C is a function of angle of incidence at the point C, the refractive index and the ratio of the densities and in this case is given by (Ewing and Press, p. 79).

$$\tan\frac{\delta}{2} = \frac{\mu_2 \tan\theta_2}{i\,\mu_1(\theta, H)\tan\theta_C},\; i^2 = -1 \qquad (16)$$

where

$$\mu_1(\theta, H) = [\mu_{V_0}^2 e^{2(p+m)H}.\sin^2\theta_0 + \mu_{H_0}^2.(H).\cos^2\theta_0\, e^{2\cdot pH}]^{1/2} \qquad (17)$$

$$\tan\theta_2 = i\,\sqrt{(1 - c^2/\beta_2^2)} \qquad (18)$$

$$\tan\theta_C = \frac{P\beta_{V_0}\cdot e^{mH}}{\xi[1 - \beta_{V_0}^2\cdot P^2\cdot e^{mH}]^{1/2}} \qquad (19)$$

By substituting the values from (17), (18) and (19) in (16), the phase change due to the reflection at the point C is given by

$$\delta = 2\tan^{-1}\left[\frac{\mu_2(1 - c^2/\beta_2^2)^{1/2}\cdot(a^2 - e^{mH})^{1/2}\xi}{e^{mH}\cdot(\mu_{V_0}^2\cdot e^{2(p+m)H}\cdot\sin^2\theta_0 + \mu_{H_0}^2\cdot e^{2pH}\cos^2\theta_0)^{1/2}}\right] \qquad (20)$$

The dispersion equation in the neighbourhood of B is obtained by putting the values of ϕ_{BCL} and δ from (15) and (20) in (6) and it is

$$\frac{K_0}{m}I - 2\cdot\tan^{-1}\left[\frac{\mu_2\,\xi(1 - c^2/\beta_2^2)^{1/2}\cdot(a^2 - e^{mH})^{1/2}}{[\mu_{V_0}^2\, e^{2(p+m)H}\sin^2\theta_0 + \mu_{H_0}^2\cdot\cos^2\theta_0\, e^{2pH}]^{1/2} e^{mH}}\right] = 2n\pi$$

where $n = 0, 1, 2, \ldots$ and I is given by (15).
Therefore,

$$kH = \frac{2mH}{\psi I}\left\{\tan^{-1}\left[\frac{\mu_2(1 - c^2/\beta_2^2)^{1/2}}{[\mu_{V_0}^2 e^{2(p+m)H}\sin^2\theta_0 + \mu_{H_0}^2\cdot\cos^2\theta_0\cdot e^{2\cdot pH}]^{1/2}} \times \frac{\xi(a^2 - e^{mH})^{1/2}}{e^{mH}}\right] + n\pi\right\} \qquad (21)$$

Putting $m = 2p$ in (1) we get the variations as

$$\mu_V = \mu_{V_0} x \exp(3pz),\; \mu_H = \mu_{H_0}\exp(pz),\; \rho(z) = \rho_0\exp(pz).$$

If we suppose that p is very small and, expand the exponential functions in power series, we have, omitting the terms of p of order higher than the first,

$$\mu_V = \mu_{V_0}(1 + 3pz),\; \mu_H = \mu_{H_0}(1 + pz),\; \rho(z) = \rho_0(1 + pz)$$

TABLE 1

ψ	Z_L	I	ψ	Z_L	I
1.01	.02569652178	.02612958781	1.16	.3298075005	.4164480186
1.02	.05084550808	.05255057676	1.18	.3575758662	.4633825646
1.04	.09935075178	.1059589848	1.20	.3825148022	.50847395
1.06	.1452386324	.1596427868	1.22	.4047962965	.5516817466
1.08	.1882817555	.2130797099	1.24	.4246089577	.5930064856
1.10	.228315666	.2886659822	1.32	.4830266707	.7404125396
1.12	.2652515052	.3174772497	1.40	.516384791	.862911025
1.14	.2990664427	.3677619383	1.42	.5218719243	.8902868596

TABLE 2

ψ	Z_L	I	ψ	Z_L	I
1.01	.03526951908	.03594395856	1.12	.3599225753	.454313434
1.02	.068995614	.07269734426	1.14	.4012270004	.5272438753
1.04	.1372660184	.1480888134	1.16	.4366948602	.5974618036
1.06	.2007270046	.2250427019	1.18	.4666700093	.664702578
1.08	.2594180944	.3024146229	1.20	.4915999494	.7398011623
1.10	.3126200334	.379139513	1.22	.5119716675	.7899906513
1.24	.528267504	.8481816053			

Under these assumptions the dispersion equation reduces to

$$kH = \frac{4pH}{\psi I}\left\{\tan^{-1}\left[\frac{\mu_2\,\xi\left(1-\frac{\beta_{H_0}^2\cdot\psi^2}{\beta_2}\right)^{1/2}\left[\frac{1}{\beta_{V_0}^2 p^2}-(1+2pH)\right]^{1/2}}{(1+3pH)\,[\mu_{V_0}^2\,(1+4pH)\sin^2\theta_0+\mu_{H_0}^2\cos^2\theta_0]^{1/2}}\right]+n\pi\right\} \tag{22}$$

Now if in (1), we take p to be so small that its terms containing powers higher than the first are neglected, we get the corresponding problem studied by Negi and Singh (1972). If in the work of Negi and Singh (1972) we work out all the steps by taking α to be so small that its squares and higher powers are neglected, we arrive at (22).

Two different models are considered. The values of c/β_{H_0} in terms of kH are calculated from equation (21) to draw the dispersion curves for fundamental and first modes and are given in Tables 3 and 4. These show that C/β_{H_0} decreases as kH increases.

TABLE 3

C/β_{H_0}	kH for n=0	kH for n=1
1.01	1598.669291	1665.332155
1.04	3111.0459916	327.0109125
1.08	122.0746879	129.7195921
1.12	66.52551294	71.47325747
1.16	41.63401387	45.27583918
1.20	28.05362499	30.93691246
1.24	19.68255981	22.07508639
1.32	9.930597606	11.73067095
1.40	3.712086807	5.168363322
1.42	1.897502986	3.288719557

TABLE 4

C/β_{H0}	kH for n=0	kH for n=1
1.01	1099.721759	1148.182564
1.04	192.9930509	204.4161069
1.08	66.7882376	72.1747958
1.12	31.83528994	35.29280673
1.16	16.88927592	19.42773265
1.18	12.33801017	14.58100633
1.20	8.663531941	10.64524898
1.22	5.828210341	7.653602232
1.24	2.91391193	4.58664778

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