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EFFECT OF AXIAL FORCE ON BEAMS WITH SHEAR AND ROTARY INERTIA EFFECTS

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INTRODUCTION

The classical solution obtained for transverse vibration of beams indicate considerable error for short beams and where higher modes are required. Rayleigh [1] introduced the concept of rotary inertia which however did not explain the higher modes until Timoshenko [2] proposed a more accurate differential equation which included both rotary inertia and shear deformation.

Much investigation has since been conducted on Timoshenko beam. A review of it is available in Ref. 3. For certain types of structures axial forces induced may impart some effect on the lateral vibration of bars. Examples of such cases are weight of the tank of the water-tower supported on columns and a rotating turbine blade—axial force being induced due to centrifugal force of rotation. No study has yet been made in general terms on this aspect. The purpose of the paper is to present a general solution for Timoshenko beam-columns. Exact solution for the particular case with hinged ends has been worked out.

EQUATIONS OF MOTION

The following equations are obtained for Timoshenko beam when the axial compression F is taken into account

$$\frac{\partial V'}{\partial x}-\rho\frac{\partial^2 Y}{\partial t^2}=0 \quad \ldots(1)$$

$$-\frac{\partial M'}{\partial x}+V'-I_0\frac{\partial^2 B'}{\partial t^2}+F\frac{\partial Y}{\partial x}=0 \quad \ldots(2)$$

$$\frac{\partial B'}{\partial x}=-\frac{M'}{EI} \quad \ldots(3)$$

$$V'=KAG\,\theta' \quad \ldots(4)$$

$$\frac{\partial Y}{\partial x}=\theta'+B' \quad \ldots(5)$$

where

$Y(x, t)$ = total deflection of the beam
$M'(x, t)$ = bending moment
$V'(x, t)$ = shear force
$B'(x, t)$ = neutral axis slope due to bending
$\theta'(x, t)$ = neutral axis slope due to shear
E = modulus of elasticity of the material of the beam
ρ = mass per unit length
I = moment of inertia of the cross-section
I_0 = ρI

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K=shape factor in shear

F=axial force, positive when compression

A=cross-sectional area

G=modulus of rigidity

Equations (1) to (5), when combined, result in the following differential equation for uniform beams

$$EI\frac{\partial^4 Y}{\partial x^4}-\left(I_0+\frac{EI\rho}{KAG}\right)\frac{\partial^4 Y}{\partial x^2\partial t^2}+\frac{I_0\rho}{KAG}\frac{\partial^4 Y}{\partial t^4}+\rho\frac{\partial^2 Y}{\partial t^2}+F\frac{\partial^2 Y}{\partial x^2}=0 \qquad \ldots(6)$$

Solution of differential equation (6) is extremely involved. So recourse had to be taken to a simpler approach suggested by Dolph [4].

SEPARATION OF VARIABLES

The following solutions are assumed for Eq. (1) to (5).

$$Y(x,t)=y(x)\,e^{iwt} \qquad \ldots(7)$$

$$V'(x,t)=V(x)\,e^{iwt} \qquad \ldots(8)$$

$$M'(x,t)=M(x)\,e^{iwt} \qquad \ldots(9)$$

$$B'(x,t)=B(x)\,e^{iwt} \qquad \ldots(10)$$

Eq. (1) to Eq. (5) reduces to the following after eliminating θ' and combining with Eq. (7) to Eq. (10)

$$\frac{dV}{dx}=-\rho w^2 y \qquad \ldots(11)$$

$$-\frac{dM}{dx}+V+I_0\,w^2B+F\frac{dy}{dx}=0 \qquad \ldots(12)$$

$$\frac{dB}{dx}=-\frac{M}{EI} \qquad \ldots(13)$$

$$\frac{dy}{dx}-B-\frac{V}{KAG}=0 \qquad \ldots(14)$$

GENERAL SOLUTION

General solution is obtained by the standard procedure. The following solutions are assumed.

$$y(x)=y_0\,e^{px} \qquad \ldots(15)$$

$$M(x)=M_0\,e^{px} \qquad \ldots(16)$$

$$V(x)=V_0\,e^{px} \qquad \ldots(17)$$

$$B(x)=V_0\,e^{px} \qquad \ldots(18)$$

where y_0, M_0, V_0 and B_0 are constants. Substituting Eq. (15) to Eq. (18) into Eq. (11) to Eq. (14),

$$pV_0+\rho w^2 y_0=0 \qquad \ldots(19)$$

$$-pM_0+V_0+I_0w^2B_0+pFy_0=0 \qquad \ldots(20)$$

$$pB_0+\frac{M_0}{EI}=0 \quad \ldots(21)$$

$$py_0-B_0-\frac{V_0}{KAG}=0 \quad \ldots(22)$$

A non-trivial solution of Eq. (19) to Eq. (22) is possible only if the determinant formed by the coefficients is zero. Thus,

$$\begin{vmatrix} p & \rho w^2 & 0 & 0 \\ 1 & pF & -p & I_0 w^2 \\ 0 & 0 & \frac{1}{EI} & p \\ -\frac{1}{KAG} & p & 0 & -1 \end{vmatrix}=0 \quad \ldots(23)$$

which yields

$$p^4+\left[\left(\frac{I_0}{EI}+\frac{\rho}{KAG}\right)w^2+\frac{F}{EI}\right]p^2-\frac{\rho w^2}{EI}+\frac{\rho I_0 w^4}{EIKAG}=0 \quad \ldots(24)$$

Solution of Eq. (24) gives

$$2p^2=-\left(\frac{I_0}{EI}+\frac{\rho}{KAG}\right)w^2-\frac{F}{EI}+\left[\left\{\left(\frac{I_0}{EI}+\frac{\rho}{KAG}\right)w^2+\frac{F}{EI}\right\}^2+\frac{4\rho w^2}{EI}-\frac{4\rho I_0 w^4}{EIKAG}\right]^{1/2} \quad \ldots(25)$$

There are four values of p

$$p_1,\ p_2=\pm\gamma$$
$$p_1,\ p_2=\pm i\epsilon \quad \ldots(26)$$

The general solutions remain same as for beams without axial load

$$y=C_1\cosh\gamma x+C_2\sin\gamma x+C_3\cos\epsilon x+C_4\sin\epsilon x \quad \ldots(27)$$

$$V=\frac{\rho w^2}{\gamma}(C_1\sinh\gamma x+C_2\cosh\gamma x)+\frac{\rho w^2}{\epsilon}(C_3\sin\epsilon x-C_4\cos\epsilon x) \quad \ldots(28)$$

$$M=EI\left(\frac{\rho w^2}{KAG}+\gamma^2\right)(C_1\cosh\gamma x+C_2\sinh\gamma x)+EI\left(\frac{\rho w^2}{KAG}-\epsilon^2\right)(C_3\cos\epsilon x+C_4\sin\epsilon x) \quad \ldots(29)$$

$$B=\frac{\frac{\rho w^2}{KAG}+\gamma^2}{\gamma}(C_1\sinh\gamma x+C_2\cosh\gamma x)+\frac{\frac{\rho w^2}{KAG}+\epsilon^2}{\epsilon}(C_3\sin\epsilon x-C_4\cos\epsilon x) \quad \ldots(30)$$

APPLICATION TO SIMPLY-SUPPORTED BEAM

As general solutions remain same for beams with and without axial force, it can be shown that after applying boundary conditions for the simply supported beam

$$\epsilon=\frac{n\pi}{L}\quad(\text{where } n=1, 2, 3, \ldots)$$

The values of w which correspond to $\epsilon=\frac{n\pi}{L}$ are obtained by setting $p=i\epsilon$ in Eq. (24). Thus

$$w^2=\frac{\left[\frac{\rho L^2}{EI}+\left(\frac{I_0}{EI}+\frac{\rho}{KAG}\right)n^2\pi^2\right]\pm\left[\frac{\rho L^2}{EI}+\left(\frac{I_0}{EI}+\frac{\rho}{KAG}\right)n^2\pi^2-4\,\frac{\rho I_0 n^4\pi^4}{EIKAG}+\frac{4F\rho I_0 n^2\pi^2L^2}{(EI)^2KAG}\right]^{1/2}}{\frac{2\rho L^2 I_0}{EIKAG}} \quad \ldots(31)$$

Eq. (31) will yield the natural frequencies of Timoshenko beam-column.

DISCUSSION

The effect of different parameters such as axial force, shear deformation, rotary inertia are best studied by maens of non-dimensional parameters. Timosehnko beam has already been studied by non-dimensional parameters S, N and λ mentioned below [4]. For Timoshenko beam-column, a new non-dimensional parameter T is introduced.

Eq. (31) may be written in the following form

$$\lambda=\frac{[1+S(1+N)n^2\pi^2\pm(1+S(1+N)n^2\pi^2)^2-4(n^4\pi^4-Tn^2\pi^2)S^2N^{1/2}]}{2S^2N} \quad \ldots(32)$$

where

$$S=\frac{EI}{KAGL^2}$$

$$N=\frac{KG}{E}$$

$$\lambda=wL^2\left(\frac{\rho}{EI}\right)^{1/2}$$

$$T=\frac{FL^2}{EI}$$

If $T=0$, Eq. (32) will reduce to Timo henko beam natural frequency. If the effects of shear and rotary inertia are neglected, then $S=N=0$ and natural frequency for such beam may be written as

$$\lambda_c=n^2\pi^2\left[1-\frac{T}{n^2\pi^2}\right]^{1/2}. \quad \ldots(33)$$

If λ_w is the natural frequency of Timoshenko beam, effect of axially compression on natural frequency is best studied by a plot of λ/λ_w against T. Such a plot for first three modes has been presented in Fig. 1. For T less than 0.1, λ/λ_w is nearly equal to 1 which means axial compression does not have any influence on the natural frequency of such bars. For higher values of T, most pronounced effect is observed in first mode and it goes on reducing with higher modes. The values of λ/λ_w mentioned in Fig. 1 is practically independent of S or N. It has been found that these values change only marginally when S is changed from 1/10 to 1/1000 and N from 1.135 to 0.3. This observation is confirmed from Fig. 2 and Fig. 3 where λ/λ_c seemed to be nearly constant for the entire range of T. It is apparent from the definition that T increases with an increase of axial compression or increase in length and decreases with an increase of flexural rigidity. This parameter describes the relative importance of axial compression for beams of a given flexural rigidity and of given length. It also brings out the fact that both length and flexural rigidity are important parameters for Timoshenko beam-column.

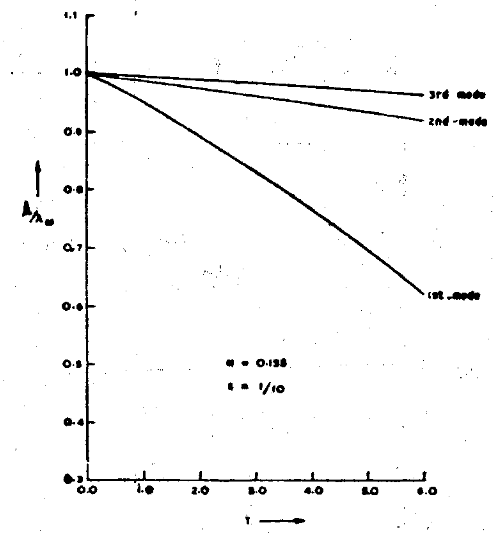

Fig. 1 Effect of Axial Compression on Natural Frequency.

Thus it is seen that reduction in natural frequency due to presence of axial load expressed as a fraction of natural frequency of beams without axial load is nearly same for both classical beams and Timoshenko beams. If this is considered to be valid for other end conditions, then it can be seen that axial compression has a more pronounced

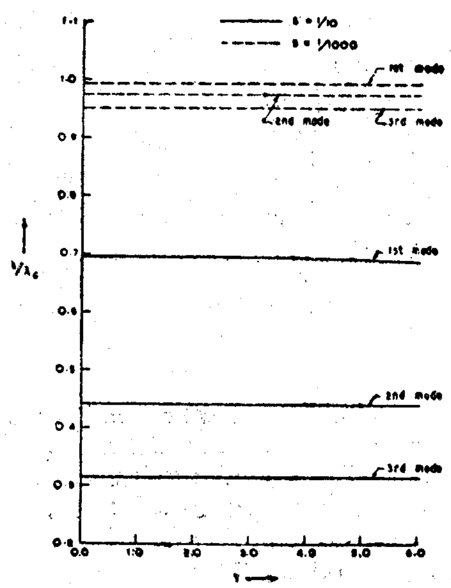

Fig. 2 Shear and Rotary Inertia Fffects on Frequency for N=0.3.

effect on a cantilever beam which is a more practical case, than a simply-supported beam. For a classical cantilever beam with axial compression acting at the ends, the natural frequency can be written as

$$\lambda_c = b\sqrt{b^2 - T} \qquad \ldots (34)$$

where $b = 1.875$ for first mode, 4.694 for second mode and so on. Definition of non-dimensional quantities λ_c and T remain same. For $T = 0.6$, the ratio of the natural frequency of the cantilever beam with axial load to that of the beam without axial load for first mode is 0.909. For hinged beams this ratio is 0.962.

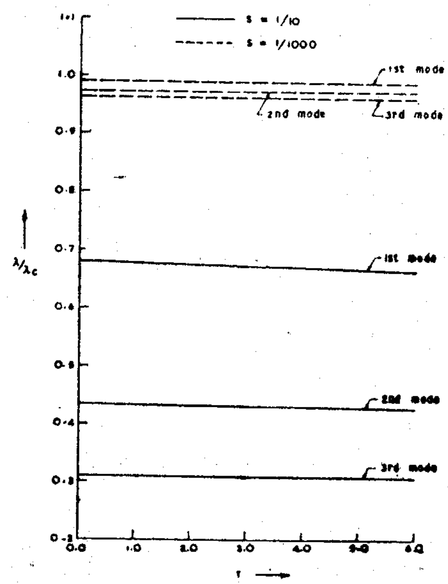

Fig. 3 Shear and Rotary Inertia Effects on Frequency for $N = 0.135$.

CONCLUSIONS

General solution for Timoshenko beam-column has been derived. Solution for natural frequencies for beams with hinged ends has been presented. Effect of axial compression has been studied from a plot λ/λ_w against T. It has been shown that for T less than 0.1, axial compression has practically no effect on the natural frequency due to lateral vibrations. For higher values of T, most significant effect of axial compression is observed in first mode. Shear deformation and rotary inertia do not seem to have any notable influence on λ/λ_w. The effect of axial compression is more pronounced for cantilever beams than for beams with hinged ends.

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