Paper No. 30

COMPARATIVE STUDY OF PREDICTING NATURAL FREQUENCY OF FOUNDATION-SOIL SYSTEM

A. Sridharan* and M.R. Madhav**

SYNOPSIS

Design of machine foundations where the loads are repetitional is complex and is commonly met with by foundation engineers. Prediction of natural frequency of the foundation-soil system is necessary since a major criterion is to design the foundation such that its natural frequency lies outside the range of the operating frequency. The many factors influencing the natural frequency are listed and a review of the well known methods of predicting natural frequency presented. The influence of the various parameters and relative merits of individual methods are discussed. Ford and Haddow's method is recommended for its simplicity and Sung's for its rigorousness.

INTRODUCTION

Design of foundations for machines where the loads are repetitional in nature, is one of the usual problems met with by the foundation engineer. This problem is much more complex than the design of foundations which support only static loads. Large machines are usually supported directly on the soil in a manner that permits a direct transmission of these periodic impulses into the soil, involving the study of soil dynamics. Review of literature shows that considerable importance is being given to the study of vibrations of massive foundations and to the study of exciting loads created by various machines since they are of great importance in engineering practices. Vibrations of machine foundations are harmful to the operation of machine itself and have harmful effect on the foundations themselves.

When out of balance machines are mounted on foundations built on the ground, the problem arises as to whether resonance of the foundations can occur. When the natural frequency of the soil foundation system coincides with the operational speed of the machine, excessive vibration amplitudes might occur leading to structural damage or even failure of the foundation.

It has been found that the natural frequencies of most foundation-soil systems are less

* Lecturer in Civil and Hydraulic Engineering Department, Indian Institute of Science, Bangalore—12.

**Burmah Shell Research Fellow in Civil and Hydraulic Engineering Department, Institute of Science, Banglore—12.

than 30 c.p.s., consequently there is the possibility of resonance occurring if the operating frequency of a machine is less than 1800 r.p.m. Many machines, especially, stationary reciprocating machines, have rotational speeds less than 1800 r.p.m. Prevention of resonance in the machine foundation system is thus one of the major criteria in their design. Therefore it is often required to design the machine foundation in such a way that the natural frequency will lie outside the range of the operating frequency of the machine.

Most of the published literature is centred round the vertical vibration of the foundation, generated either by a single blow resulting in free oscillations, or by continuously applied sinusoidal force giving forced oscillations. Vibrations so generated are the ones most commonly met with in practice.

The earliest approach to the problem of foundation vibration was made by Degebo (1) by considering the vibrating system to behave as a single mass supported by a weightless spring. As a result of their extensive series of tests, it was necessary to consider the spring to have an effective mass.

The second approach was to consider the problem as the vertical motion of an oscillator resting on a semi-infinite isotropic, homogeneous, elastic body. Most of the existing works are based on either of these two approaches. Interesting review can be found in the publications of Weil (1963) and Prakash and Bhatia (1964). It is the purpose of this paper to examine and compare some of the well known works and discuss the relative importance of various parameters affecting the natural frequency of foundation soil systems.

FACTORS INFLUENCING THE NATURAL FREQUENCY OF FOUNDATION SOIL SYSTEM

There are too many factors which influence the natural frequency of foundation soil system to list them all, but the most important are (i) the static load which the foundation carries (ii) the shape and size of the foundation (iii) the exiting force (for forced oscillation), (iv) the type of contact pressure distribution (v) the depth of embedment of foundation and (vi) the soil type.

THE SOIL SPRING ANALOGY

The first approximation made in the study of foundation vibrations was by considering the system as a single mass supported by a weightless spring. Later, it was proposed that a certain soil mass underlying the foundation should be considered to oscillate together with the machine foundation, in which case the natural frequency 'f_n' is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k.g}{W_v + W_s}} \tag{1a}$$

or

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k'Ag}{W_v + W_s}} \tag{1b}$$

where, A is area of contact

k dynamic Spring constan

k′ dynamic soil modulus

W_s weight of soil actively participating with the oscillation or equivalent weight of soil. (It has no clear physical boundaries).

W_v total static load.

g acceleration due to gravity

This leaves the designer the difficult task of estimating the dynamic soil spring constant k′ and the value of W_s. No satisfactory method has so far been developed for the determination of these soil coefficients.

Tschebotarioff and Ward (1948) modified the equation (1) and introduced a new parameter 'Reduced Natural Frequency' in order that the modified equation could be useful in comparing foundation of different areas and subject to different pressures. An ampirical relationship was established between the area of contact of the foundation and the parameter 'reduced natural frequeney' denoted as f_{nr} which bears the following relationship with f_n :

$$f_{nr} = f_n \times \sqrt{p} \tag{2a}$$

where p is the intensity of load.

$$\text{Hence } f_n = \frac{1}{2\pi}\sqrt{\frac{A}{W_v}}\cdot\sqrt{\frac{k' g}{(1+W_s/W_v)}} \tag{2b}$$

In a later publication Tschebotarioff (1953) further confirmed the relationship between area and f_{nr} with some more data on the performance of full scale foundations. The relationship between f_{nr} and area is shown in Fig. (1).

Number of limitations can be cited for the above work. Tschebotarioff himself has listed down some of them. The resulting lines in Fig. (1) are based on a limited number of performance records and several inconsistencies can be noticed. Furthermore, both horizontal and vertical unbalanced forces and their combinations were treated together. The ratios of height to width of the foundations varied quite widely, and the natural frequencies determined by several different methods. The plotting of results on a log-log scale is likely to mask the actual behaviour The f_{nr} vs. area relationship is independent of static intensity, shape of the area of foundation, depth of embedment and the exciting force (for forced oscillations).

All these factors considered above are bound to affect the natural frequency and possibly some of the factors may cancel with each other and some may be cumulative. However, keeping constantly in mind, all the limitations and uncertainties involved the empirical relationship between f_{nr} and area established may still serve as a guide to help the engineer reasonably, especially since the relationship is from valuable field records.

PAUW'S METHOD

Pauw (1953) assumes the stresses to be distributed uniformly and in an effective zone

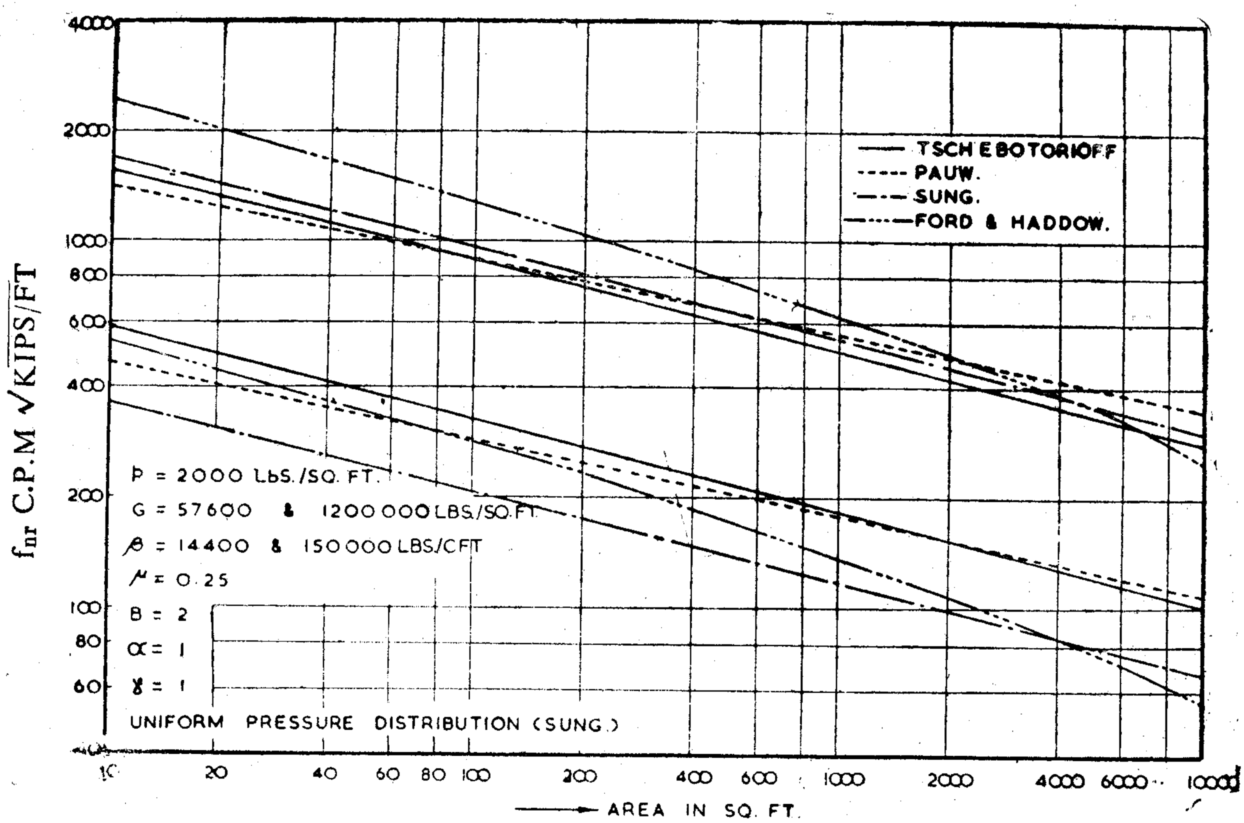


Figure 1

defined by a truncated cone or pyramid, the sides of which slope away at an angle whose tangent is $\alpha/2$. Further the modulus of elasticity is assumed to be constant for cohesive soils and proportional to the effective depth for cohesionless soils. Thus

$$E_z = \beta (h + z) \qquad \text{(for cohesionless soils)} \qquad (3a)$$

and $$E_z = E_0 \qquad \text{(for cohesive soils)} \qquad (3b)$$

where, E_z is the modulus at any depth z,

β rate of increase of modulus with depth,

h equivalent height $= \dfrac{\text{Intensity of static load}}{\text{unit weight of soil}}$

E_0 modulus at the surface.

With these assumptions the spring constant is determined by integrating, over the depth of the pyramid, the displacement of each infinitesimal layer due to a given load on the foundation.

The equation for equivalent weight of soil oscillating with vibration has been derived by equating the kinetic energy of the affected zone to the kinetic energy of a mass assumed to be concentrated at the base of the foundation. Substituting the values for spring constant and equivalent weight of soil in equation (1a) the natural frequency can be found. The various parameters considered by Pauw are (1) shape and size of foundation, (2) the load dispersion angle or the slope of the pyramid, (3) the density of the soil, (4) the soil type represented by β and (5) the intensity of static loading. The use of this approach does not require the soil to be homogeneous.

In order to use the above method one must know the rate of increase with depth and value at the surface of the modulus of elasticity and the value of α. No satisfactory method has been recommended to obtain the value of β and α. According to this anology the value of 'k' increases with the static intensity which is not in conformity with the existing knowledge. The modulus of subgrade reaction 'k' (i.e. spring constant) reduces as the static load increases. Another limitation of this method would be the non-convergence of the infinite integral to a finite limit for the determination of the equivalent soil mass in the case of cohesive soil.

SUNG'S METHOD

Reissner (1936) presented an analytical solution for the vertical motion of an oscillator resting on an elastic homogeneous isotropic semi-infinite continuum. This analysis assumes the vertical periodic, pressure forces to be uniformly distributed. The resonant frequency, amplitude of oscillation and power requirements of the vibrating soil system were determined as functions of radius of loaded area, static weight of vibrator, weight and frequency of the vibrating mass, and material properties of soil.

Sung (1953) and Quinlan (1953) have extended Reissner's treatment to cover different contact pressure distributions between the oscillator and elastic body obtaining essentially the same results.

Sung (1953) has considered three different axially symmetrical pressure distributions i.e. uniform, parabolic and that produced by a rigid footing on a clayey soil. The basis of the Sung's theory is Navier's displacement equilibrium equations neglecting damping. The various parameters considered by Sung are (1) the radius γ_o of the loading area, (2) the total mass m_o, (3) the amplitude of dynamic force applied, (4) the distribution of contact pressure on the base, (5) Poissons' ratio, μ, (6) mass density ρ, and (7) shear modulus G of the foundation material. The results of analysis are presented graphically, with the dimensionless maximum amplitude and dimensionless frequency at maximum amplitude plotted against μ, the Poisson's ratio for different values of 'b' (a significant parameter referred as the mass ratio) for all the three pressure distributions analyzed.

In order to adopt Sung's method the designer must know among other things, the shear modulus G, Poisson's ratio μ and the type of pressure distribution. Indeed Sung suggests small scale dynamic tests to find the soil constants and the type of pressure distribution. An uncertainty introduced by the Sung's approach pertains to the type of pressure distribution involved which is significantly affected by the magnitude of oscillating force and the static intensity Weil (1963). Sung also makes a questionable assumption that 'the shape has no influence on 'f_n'.

FORD AND HADDOW'S METHOD

Ford and Haddow (1960) have suggested a simple method of predicting the natural frequency of machine foundation based on Raleigh's principle. In this method the maximum strain energy of the system has been equated to the maximum kinetic energy of the system to derive an expression for the natural frequency.

Further, the dynamic stress at any depth 'z' is uniformly distributed over a section of the solid, parallel to the 'x y' plane that is, parallel to the base of the foundation and the whole system has been considered as conservative.

With these assumptions and using Raleigh's principle, Ford and Haddow arrive at the expression for natural frequency.

$$f_n = \frac{1}{2\pi} \sqrt{\frac{E' \beta' ab}{\frac{\rho\, ab}{\beta'} + M}} \tag{4a}$$

i.e.
$$f_n = \frac{1}{2\pi} \sqrt{\frac{2G(1+\mu)\beta' ab}{\frac{\rho\, ab}{\beta} + M}} \tag{4b}$$

where, M = mass of the foundation and machine

ρ = mass density of the soil

a, b = two sides of the foundation block

E' = dynamic modulus of elasticity of soils

β' = is the decay factor which is defined from

$$W_o = W_{of}\, e^{-\beta' z} \tag{5}$$

where, W_{of} = amplitude of vibration at the surface,

W_o = amplitude of vibration at a depth z.

The decay factor β' is given by an expression

$$\beta' = \frac{B}{m' \sqrt{A}\,(1-\mu^2)} \tag{6}$$

where, B = a soil constant (varies between 1.5 for clays and 2.0 for sands).

m' = shape factor — a constant

A = area of foundation

μ = Poisson's ratio

To adopt this method in practice one must know the soil constants such as E' or G, μ and m' from dynamic tests.

Among other things the expression for the decay factor forms an uncertainty. How far this expression is valid for soils needs confirmation. Though uniform distribution of dynamic stress is inaccurate, any other form of assumptions will make the analysis further complex with a negligible gain in accuracy. This procedure also neglects the losses such as damping.

DISCUSSION

In the foregoing paragraphs four well known methods of predicting the natural frequency of foundation soil system have been presented. None of these methods considers all the parameters which have been listed earlier in this paper which affect the performance of an oscillator. One or more factors have been ignored by each method giving importance to other parameters. A study of the influence of each parameter as given by each method and a comparison between these methods will certainly help in better understanding the vibrations of foundation soil system.

The authors have computed the values of 'f_n' and 'f_{nr}' using the above three analytical methods, for a set of foundations of different shapes and sizes and for different values of G, μ, β, B, a, p and different types of pressure distribution, to compare with Tschebotarioff's

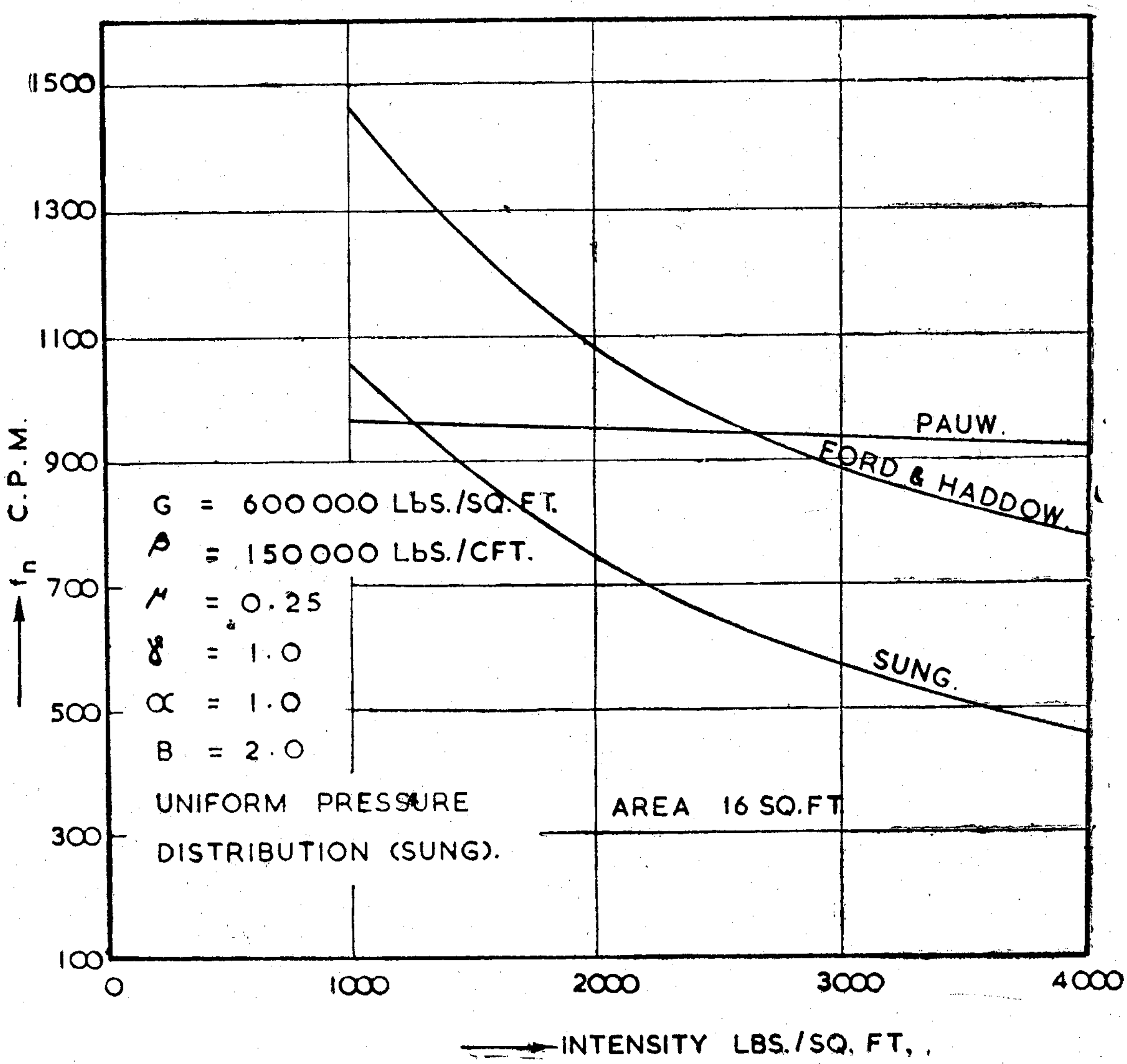


Figure 2 (a)

empirical relationship. For the purpose of comparison, suitable values for the soil constants such as G, μ, β, B and the physical properties of foundation soils have been assumed and the f_n and f_{nr} have been evaluated. These parameters have been varied quite widely and only limited results have been presented for conciseness. The values used for various parameters have been given in each figure. The following relationship between β (Pauw) and G has been made use of to have comparative values in all the three methods.

$$E_o = \beta (h) \quad \ldots \quad (Z \text{ being zero})$$
$$= 2G (1 + \mu)$$

$$\text{or } \beta \left(\frac{p}{\rho}\right) = 2G (1 + \mu)$$

$$\text{or } \beta = \frac{2G (1 + \mu) \rho}{p}$$

where ρ is the unit weight of the soil in lbs/cft.

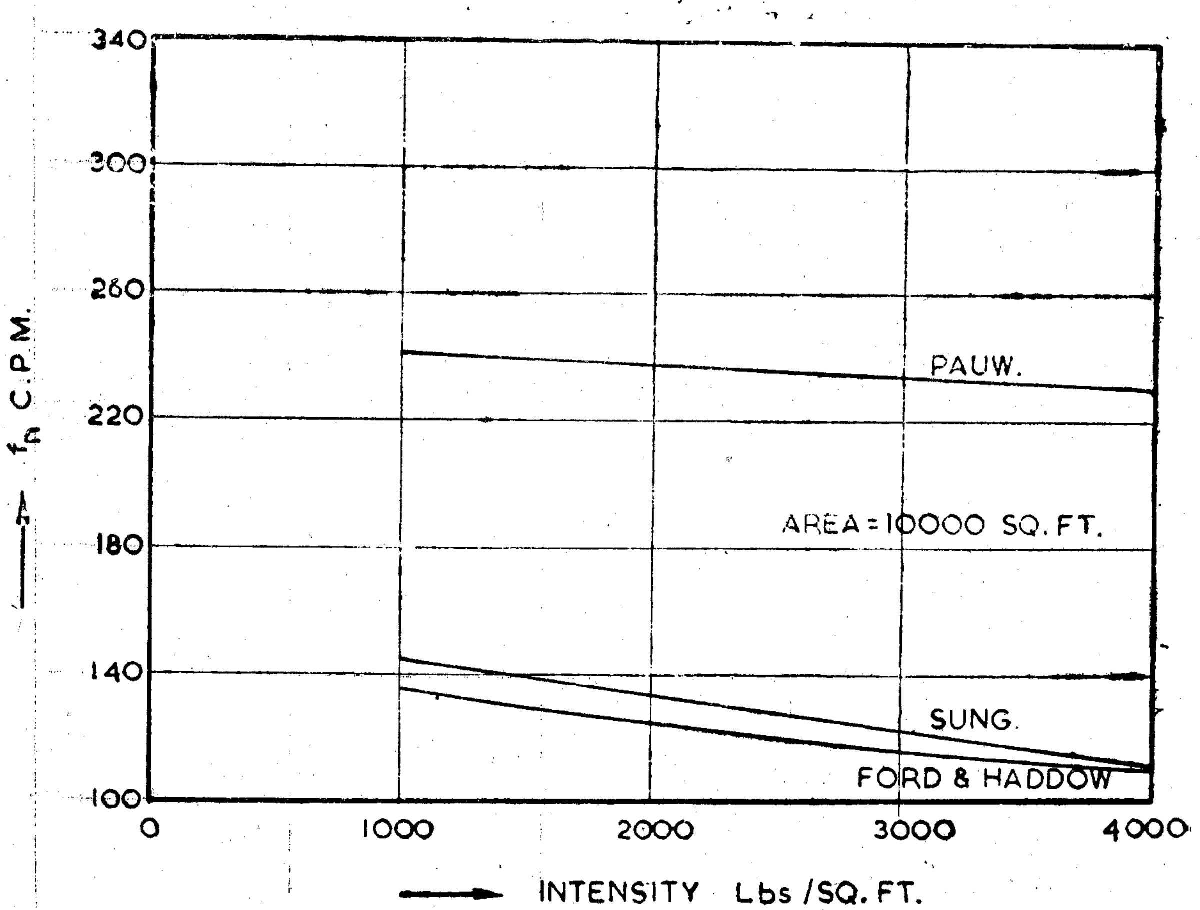


Figure 2 (b)

All the three methods i.e. of Sung, Pauw and Ford and Haddow consider the influence of laod intensity on the natural frequency. It is an accepted fact that an increase in the

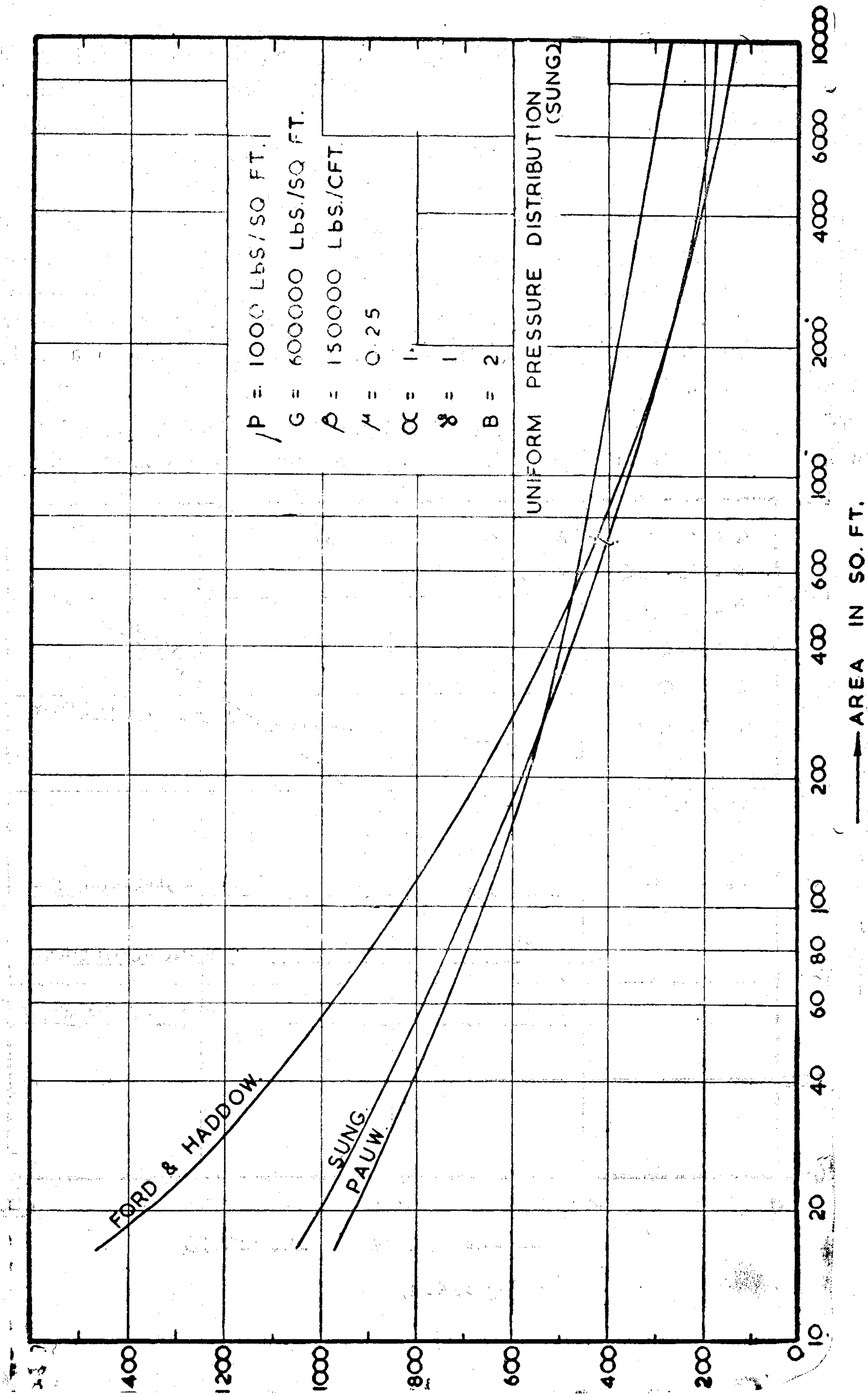

P = 1000 LbS/SQ FT.
G = 600000 LbS/SQ FT.
ρ = 150000 LbS/CFT.
μ = 0.25
α = 1
γ = 1
B = 2
UNIFORM PRESSURE DISTRIBUTION (SUNG)
FORD & HADDOW
SUNG
PAUW
AREA IN SQ. FT.
10
20
40
60
80
100
200
400
600
1000
2000
4000
6000
10000
0
200
400
600
800
1000
1200
1400

Figure 3

intensity of loading decreases the natural frequency. Many published records (East Wood 1953and Ford and Haddow 1960,) show the static intensity to have a great influence on natural frequency and that the natural frequency decreases to an extent of 50 to 60%. Fig. 2a and 2b show the relation between static intensity and the natural frequency for two extreme areas of contact. The rate of decrease of natural frequency with load intensity is almost same for Sung and Ford and Haddow methods whereas Pauw's method in contrast shows a neglegible decrease in f_n with increase in intensity. It is mainly due to the questionable assumption that 'E' increases with intensity of static loading. The effect of Area on 'f_n' for three different methods is shown in Figure 3. It can be discerned from the same that at smaller areas the difference between Sung and Ford and Haddow methods is considerable which decreases to a negligible value as area increases.

The effect of intensity on f_{nr} is considered in figures 4a, 4b, 4c and 8. Tschebatorioff's empirical plot exhibits that f_{nr} is not affected by the intensity. Sung's method also shows

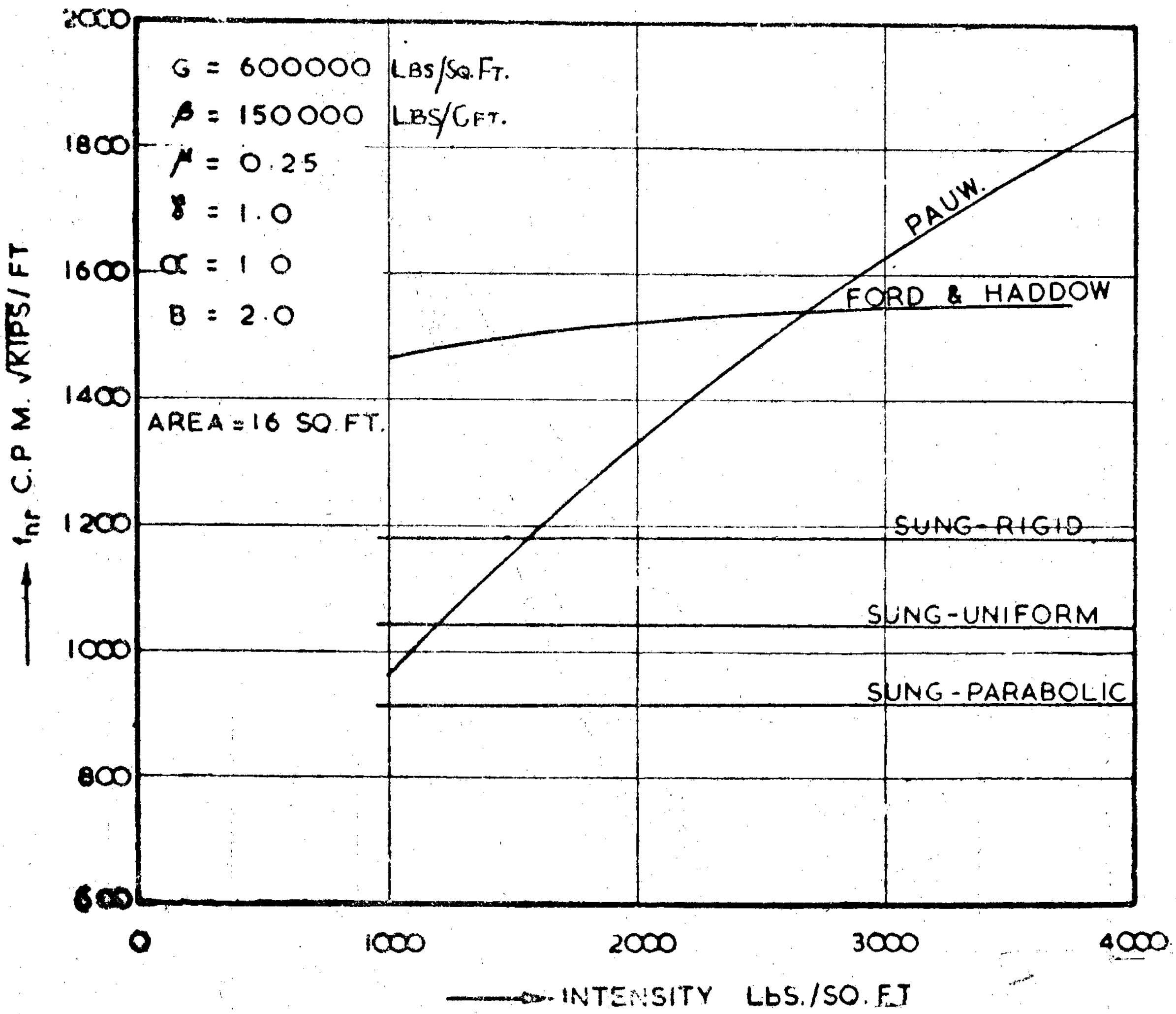


Figure 4 (a)

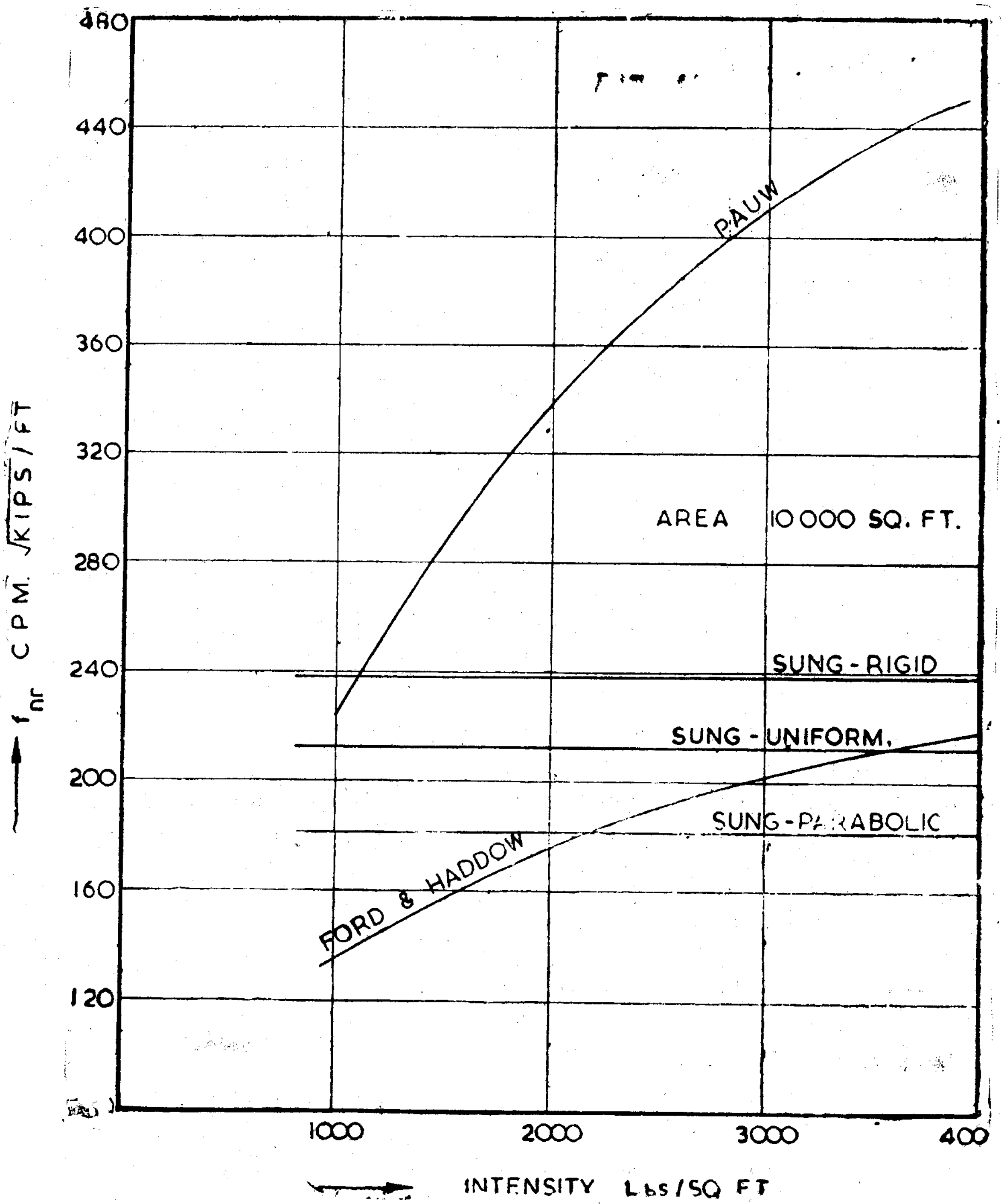


Figure 4 (b)

that the intensity has no effect on f_{nr} for all the three types of pressure distributions and Ford and Haddow's method shows a slight increase in f_{nr} with intensity at smaller areas, increasing for larger areas. The difference in f_{nr} for various intensities increases with area by Ford and Haddow's method (Fig. 4c) and this is always much less than that shown by Pauw's method

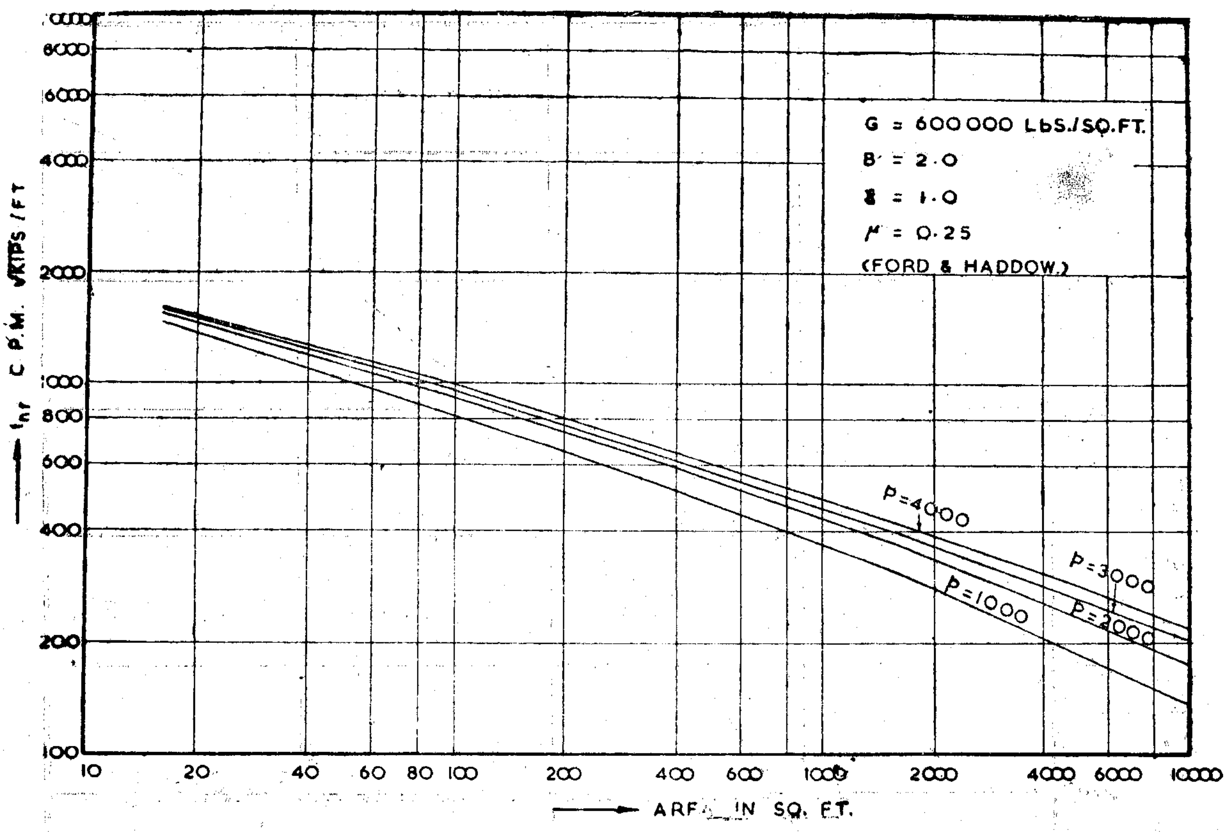


Figure 4 (c)

(Fig. 8). The f_{nr} increases significantly with intensity for all areas of contact in Pauw's method which is not consistant with Tschebotarioff's plot and the other two methods. This marked difference in the behaviour is mainly due to the basic assumption that 'E' increases with static intensity, as stated earlier.

The influence of shape has been taken into consideration by Ford and Haddow and Pauw's methods only. The other two methods ignore the shape effect.

Table 1 shows the effect of shape on f_{nr}. There is a general agreement between the Pauw and Ford and Haddow methods to the extent that the f_{nr} increases with the change in shape from circular to rectangular. The increase in f_{nr} is in the range of 6–12% being larger for larger areas. With the complexity in the behaviour of subgrades under dynamic loading and other vagaries in assessing values for the parameters, differences of less than 10% are unimportant. Hence considering all factors it can be said that the shape of the foundation block has neglegible effect on f_{nr} and f_n of the system.

Figs. 5a and 5b show the influence of μ on f_n for two different areas: The Sung and Ford and Haddow methods show consistent increase in f_n or f_{nr} and the effect of small change in μ on f_n or f_{nr} is phenomenal.

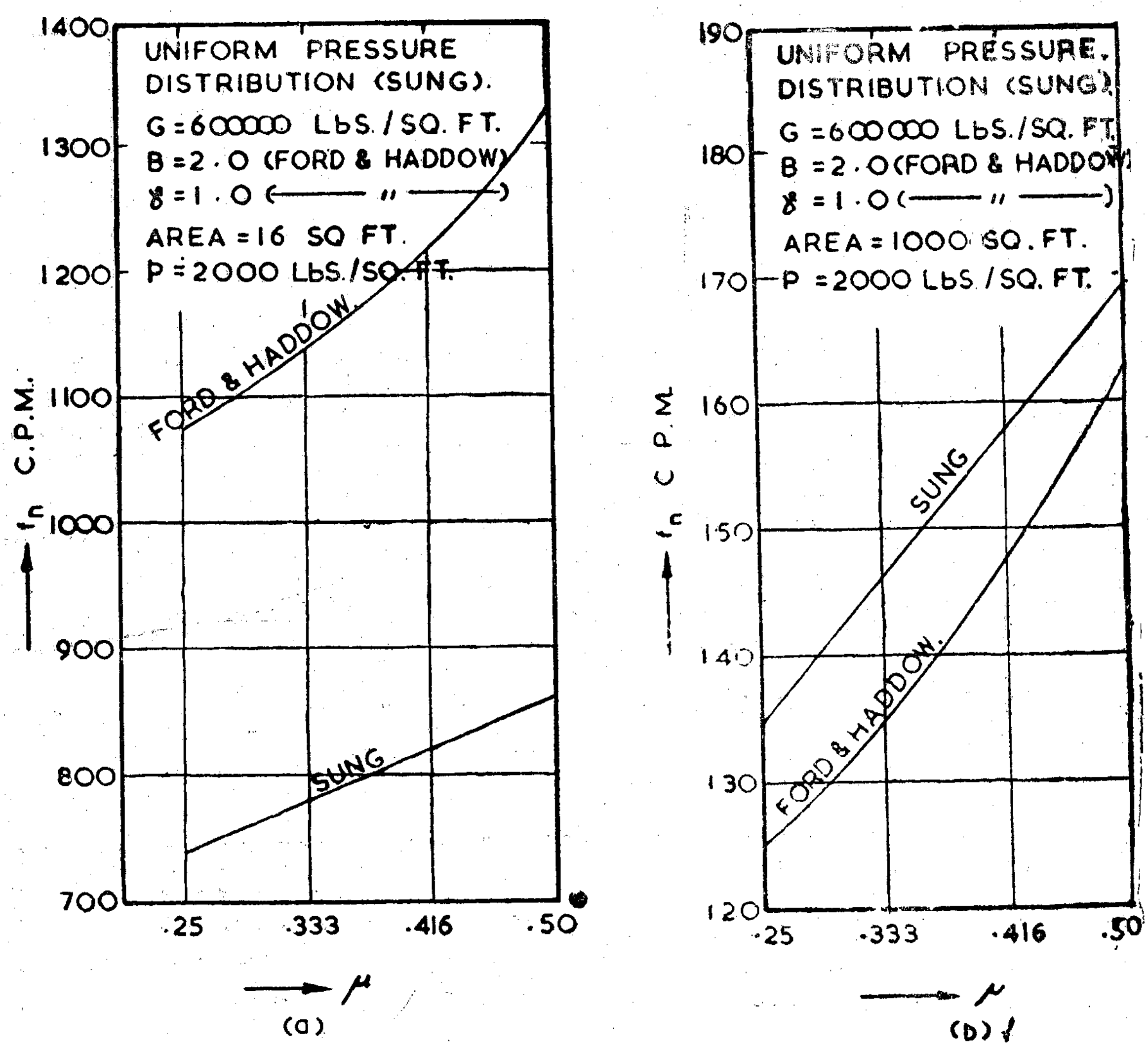
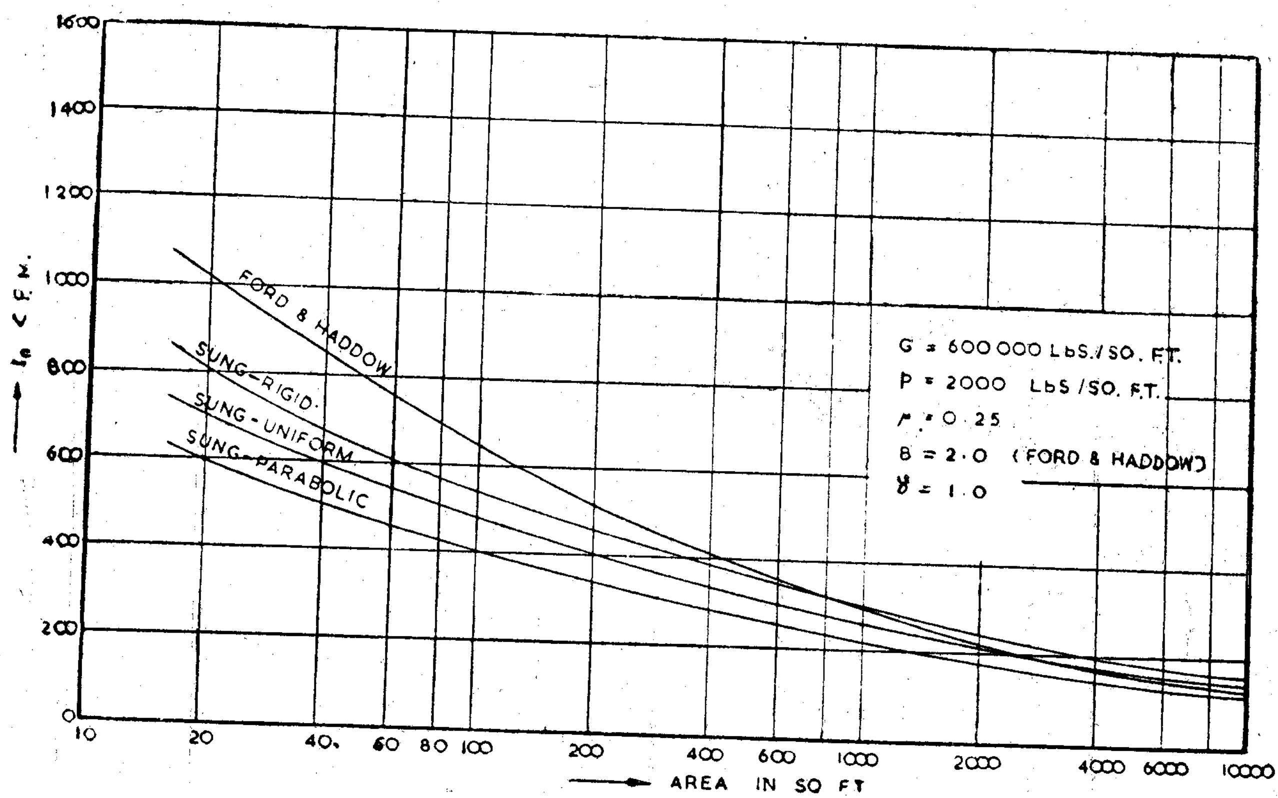


Figure 5 (a,b)

Fig. 6 demonstrates the effect of different types of pressure distribution on f_{nr}. This clearly brings out the necessity of the true knowledge about the pressure distribution if Sung's method is to be adopted. For larger areas the effect is not marked.

Fig 7 exhibits the effect of B values on f_n. The increase in the values of f_n with B is obvious and the effect of 'B' on f_n is phenomenal. Larger the value of B, the more will be the percentage increase of f_n with area. This necessitates proper judgements in the values of B.

Fig. 8 illustrates the effect of 'α' the dispersion angle, in Pauw's method on f_{nr}. A decrease in α from 1.75 to 0.875 reduces f_{nr} to an extent of about 35%. In the absence of a positive method to determine the value of α the assessment of α from experience and judgement may pose a difficult problem.



Figur 6

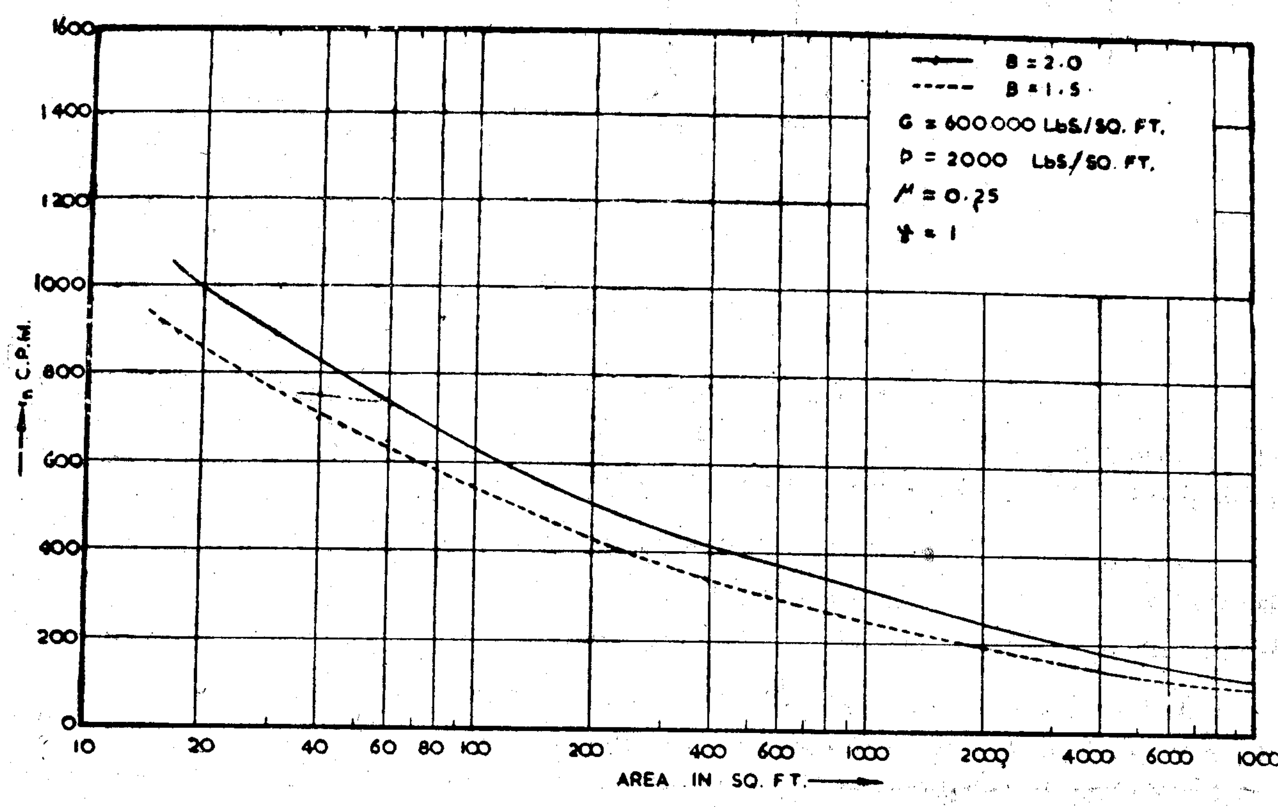


Figure 7

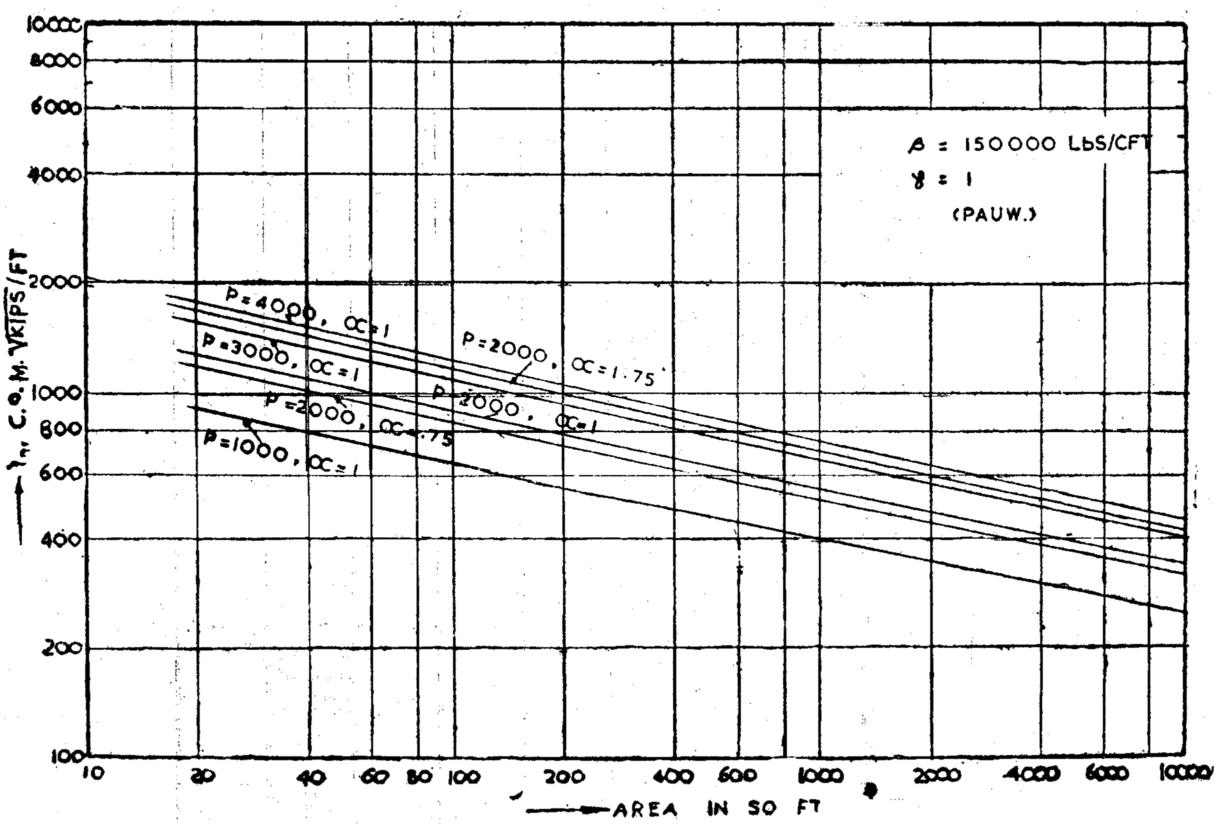


Figure 8

Fig. 9 shows the relation between $\sqrt{G}$ or $\sqrt{\beta}$ with f_{nr} for three different areas. Since f_{nr} is directly proportional to $\sqrt{G}$ or $\sqrt{\beta}$ a straight line relationship is obtained. It is rather interesting to note that all the three methods i.e. Pauw's, Sung's and Ford and Haddow's derive the expression for f_n such that the influence of p, shape, size, μ, B, α and type of pressure distribution is not at all affected by the value of G or β. In other words, for example, the rate of decrease of f_n must be same either for a clayey soil or a sandy soil irrespective of their values of G or β. Experimental confirmation is essential to this effect.

Fig. 1 shows the relation between f_{nr} and Area for two limits of G and β values, for all the three methods. Tschebotarioff's boundary lines i.e. line for sand stone and line for peats have also been included in the plot. The values of G and β were chosen such that they represent the extreme limits for soils. The values for other parameters are also given in the figure. Since the influence of these parameters has been shown earlier, the readers can easily imagine the effect of the possible changes of these parameters in shifting the lines. It is seen that there is a basic agreement between all the four methods especially with Sung's and Ford and Haddow's and Tschebotarioff's plot. The intensity of load for this plot has been taken as 2000 lbs./sq.ft. A change in the value of load intensity alters the position of Pauw's boundary significantly, Ford and Haddow's neglegibly and Sung's none at all. It is evident

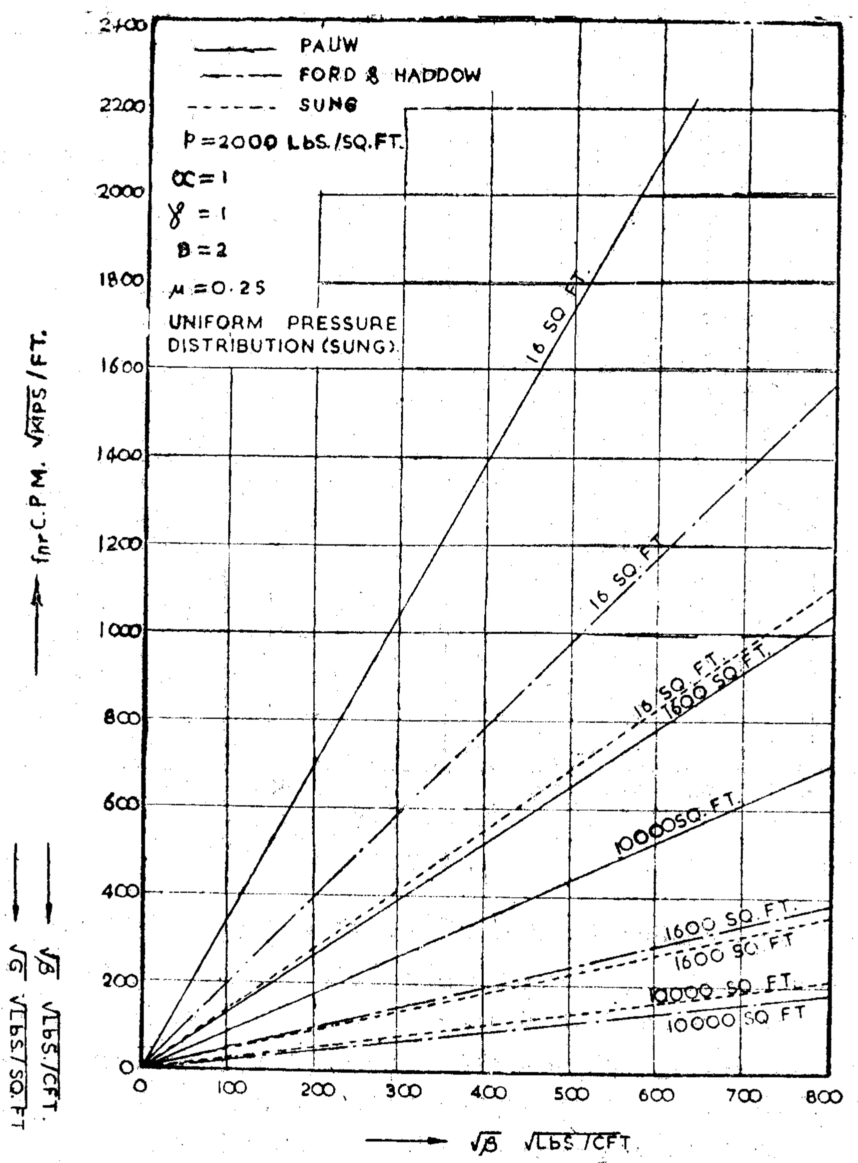


Figure 9

from this illustration that Tschebotarioff's plot almost fixes up the boundaries of danger zone for the occurrance of resonance in machine foundations.

CONCLUSIONS

From the above comparative study, the following conclusions can be drawn for foundations subjected to vertical oscillation.

With different basic approaches Sung's and Ford and Haddow's methods show a similarity with Tschebotarioff's plot between area and f_{nr}.

Ford and Haddow's method is recommended for its simplicity and Sung's for its rigourousness if the type of pressure distribution is known.

Owing to the complex factors involved in the design of foundations subjected to vertical oscillating forces it is hard to expect, any single method to satisfy all the requirements. Hence Sung's and Ford and Haddow's analyses may be applied for any problem and the probable value judged on individual merits.

ACKNOWLEDGEMENT

The authors are highly indebted to Prof. N.S. Govinda Rao, Professor, Civil and Hydraulic Engineering Department, for his keen interest and encouragement in the course of these investigations. They are grateful to Dr. B.V. Ranganatham for his helpful suggestions and criticism afforded during the preparation of the paper. The facilities provided by the Indian Institute of Science and the Fellowship to the Junior author by Burmah Shell Company are gratefully acknowledged.

REFERENCES

East Wood, W., (1953), "Vibrations in Foundations", Structural Engineer, March 1953, pp. 82–86.

Ford and Haddow (1960) "Determining the Machine Foundation Natural Frequency by Analysis", The Engineering Journal. Vol. 43, No. 12.

Lorenz, H. (1934), "New Results Obtained from Dynamic Foundation Soil Tests", National Research Council of Canada. Tech. Translation TT 521 (1955)

Pauw, A. (1953), "A Dynamic Analogy for Foundation Soil System", Symp. on Dynamic Testing of Soils ASTM., S.T.P. No. 156, pp. 90–112.

Prakash, S. and U.K. Bhatia, (1964), "A Review of Machine Foundation Behaviour", Bulletin, Indian Soc of Farthquake Tech., Vol. 1 No. 1, Jan. 1964, pp. 45–64.

Quinlan, P.M. (1953), "The Elastic Theory of Soil Dynamics" Symp. on Dynamic Testing of Soils ASTM., S.T.P. No. 156, pp. 3–34.

Reissner, E. (1936), "Stationare, axialsymmetrishe durch eine schuttelnde Masse erregte Schwingungen eines homogenen clastischen Halbraumes," Ingenieur-Archiv, Band vii, pp. 381–396.

Sridharan, A. (1962), "Settlement Studies on a Model Footing under Dynamic Loads," Proc. Second Symp. on Earthquake Engineering, University of Roorkee, Roorkee, India.

Sridharan, A. (1964), "Natural Frequency of Model Pile Foundation", Proc. Symp. of Bearing Capacity of Piles, Central Build. Res. Inst., Roorkee, India, pp. 136–141.

Sung, T.Y. (1953), "Vibrations in Semi-infinite Solids due to Periodic Surface Loading", Symp. on Dynamic Testing of Soils ASTM., S.T.P. No. 156, pp. 35–63.

Tschebotarioff, G.P. and E R. Ward (1948), "The Resonance of Machine Foundations and Soil Coefficients Which Affect it", Proc. Sec. Int. Conf S.M. & F.E., Rotterdam, Vol 1, pp. 309–313.

Tschebotarioff, G.P. (1953), "Performance Records of Engine Foundations", Symp. on Dynamic Testing of Soils, ASTM., Sp. Tech. Pub. No. 156 pp. 163–168

Weil, N.A. (1963), "On Dynamics of Machine Foundations" Proc. of the International Symposium on Measurement and Evalution of Dynamic Effects and Vibrations of Constructions RILEM, Budapest, Hungary.

TABLE 1

Values of Natural Frequency for different shapes and sizes

p = 2000 lbs./sq. ft. β = 150,000, G = 600,000 μ = .25

B = 2.0 a = 1.0

γ = Ratio of Length to Breadth.

Sl. No.	Area	Ford and Haddow				Pauw			
		Circular	$\gamma=1$	$\gamma=2$	$\gamma=4$	Circular	$\gamma=1$	$\gamma=2$	$\gamma=4$
1	16	1064	1068	1085	1128	900	910	924	940
2	100	636	644	660	686	595	605	617	631
3	400	414	419	428	448	444	452	462	474
4	1600	254	259	267	280	326	334	343	353
5	4900	166	170	173	184	263	270	278	288
6	10000	124	126	129	132	222	229	237	247

Paper No. 31

DYNAMIC RESPONSE OF RECTANGULAR PLATES ON ELASTIC FOUNDATION

P.C. Sharma*

SYNOPSIS

Navier type solution for the dynamic response problem of rectangular plates, simply supported all arround, is presented here. Case of a plate supported on Winkler type foundation and also having damping, is considered and solution obtained. The solutions thus obtained can be useful in the dynamic response analysis of more complicated cases.

INTRODUCTION

The dynamic theory of plates finds many applications in modern technology such as the analysis and design of buildings, aircrafts, ship hulls and pavements. Except for a few exceedingly simple cases, an exact mathematical analysis of such problems is practically impossible. This is even more so for the case of plates on elastic foundation which is important for example in rigid pavement design.

From engineer's standpoint, both frequency and displacement are significant quantities. Bending moment responses can be easily obtained from displacement responses. Therefore it is important to get the displacement response. Unfortunately not much is found under dynamic response of plates, even for simple cases where the solution is straight forward. Since engineers do not have time to devote to routine mathematical manipulations and derivations, it is of some significance to have these results available. Therefore the purpose of this paper is two fold:

1. To demonstrate the use of Navier's method of analysis for dynamic response problem;
2. To make available a few basic results which can be of further use in the dynamic response analysis of plates on elastic foundation.

DIFFERENTIAL EQUATION

The governing differential equation for the small deflections of an elastic thin plate subjected to a lateral loading q (x, y) is given by:

$$\nabla^4 \omega = \frac{\partial^4 \omega}{\partial x^4} + \frac{2\,\partial^4 \omega}{\partial x^2 \partial y^2} + \frac{\partial^4 \omega}{\partial y^4} = \frac{q\,(x,\,y)}{D} \qquad (1)$$

*Reader in Civil Engineering, Govt. Engineering College, Jabalpur, India.

in which w is the diflection, x, y are space coordinates and D is the flexural rigidity of the plate. (see fig. 1)

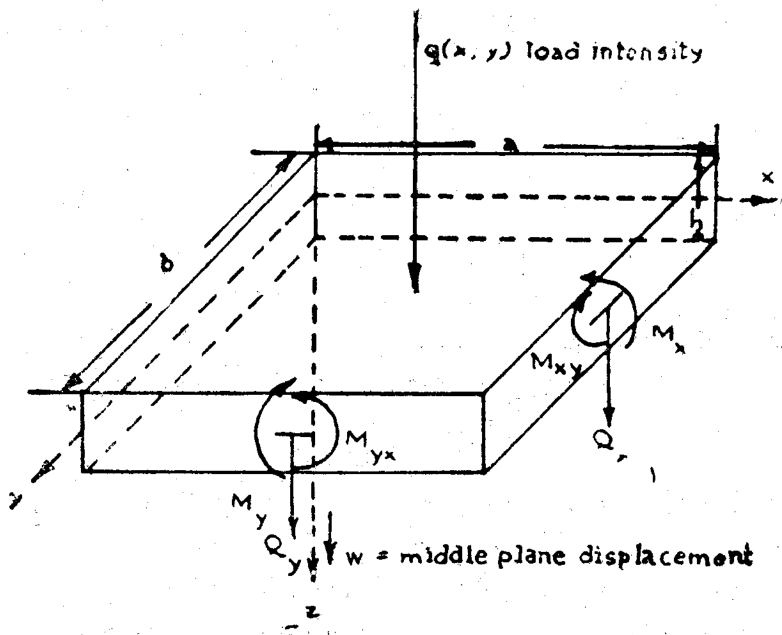


Figure 1 Plate Notation

For the case of dynamic loading and Winkler type elastic foundation with viscous damping, the quation of motion is obtained by replacing q (x.y) in Equation 1 by$(m\frac{\partial^2 w}{\partial t^2} + c\frac{\partial \omega}{\partial t} + kw) + P(x, y, t)$, where $m\frac{\partial^2 \omega}{\partial t^2}$ is the inertia force and $c\frac{\partial \omega}{\partial t} + kw$ is the reaction of the foundation including the effect of viscous damping. Also, P (x, y, t) is the forcing function. Thus Equation 1 becomes:

$$\nabla^4 \omega + \frac{kw}{D} + \frac{c}{D}\frac{\partial \omega}{\partial t} + m\frac{\partial^2 \omega}{\partial t^2} = \frac{P(x, y, t)}{D} \qquad (2)$$

This equation together with the appropriate boundary conditions governs the dynamic respo-n e of the plate system to the dynamic loading P (x, y, t).

Mathematically speaking, this partial differential equation is of the parabolic type, and is referred to as a propagation problem in two space dimensions. The solution "marches" in the time domain starting with the initial conditions and confined in space by the boundary conditions. In other words, for the case of rectangular plates, considered in this paper, the solution has to march inside a box (as shown in Fig. 2) whose base is made up of the initial conditions and all the four sides are made up of the boundary conditions, the top being open.

NAVIER TYPE SOLUTION

The double sine series solution of the problem of forced vibration of a simply suppor-

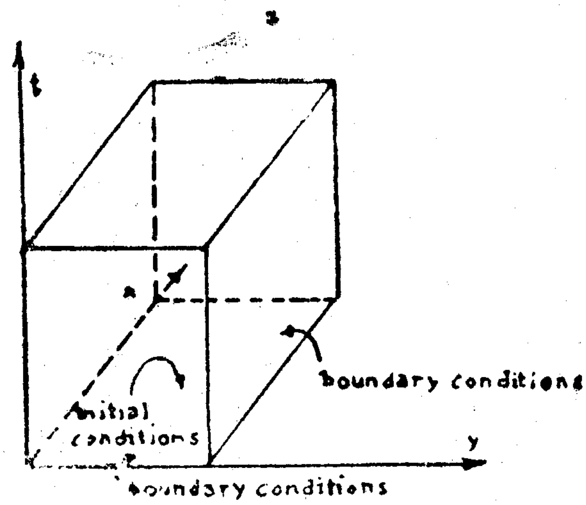


Figure 2 Pictorial Representation of Propagation Problem

ted rectangular plate on an elastic foundation is presented here, which is based on the Navier's solution for the static case. Referring to Equation 2 the solution using the technique of separation of variables, may be written as:

$$w = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} S_{ij} T_{ij} \tag{3}$$

where $S_{ij} = \text{Sin} \frac{i\pi x}{a} . \text{Sin} \frac{j\pi x}{b}$ and T_{ij} is a function of time only. Also assuring the loading function be given as:

$$P(x, y, t) = G(x, y) F(t) \tag{4}$$

where G (x, y) is a function of the space coordinates 'x' and 'y' only and F (t) is a function of time 't' only.

Let G (x, y) be expanded in a double sine series:

$$G(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g_{ij} \text{Sin} \frac{i\pi x}{a} \text{Sin} \frac{j\pi y}{b} \tag{5}$$

in which

$$g_{ij} = \frac{4}{ab} \int_0^a \int_0^b G(x, y) \text{Sin} \frac{i\pi x}{a} \text{Sin} \frac{j\pi y}{b} dxdy \tag{6}$$

Substituting the preceding expressions 3, 4 and 5 into the equation of motion (Equation 2), the following equation is obtained

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left[\left(\frac{i\pi}{a}\right)^4 S_{ij} T_{ij} + \frac{2(i\pi)^2(j\pi)^2}{ab} S_{ij} T_{ij} + \left(\frac{j\pi}{b}\right)^4 S_{ij}T_{ij} + \frac{k}{D} S_{ij}T_{ij} + \frac{c}{D} S_{ij}\dot{T}_{ij} + \frac{m}{D} S_{ij} \ddot{T}_{ij} - \frac{g_{ij}}{D} S_{ij} F(t) \right] = 0 \tag{7}$$

Since S_{ij} is not identically zero, one obtains :

$$\ddot{T}_{ij} + 2r\,\dot{T}_{ij} + p_{ij}^2\,T_{ij} - \frac{g_{ij}}{m}F(t) = 0 \qquad (8)$$

where $r = \frac{c}{2m}$ and p_{ij} is the natural undamped circular frequency of the $(i, j)^{th}$ mode of the plate :

$$p_{ij} = \sqrt{\frac{D}{m}\left[\left\{\left(\frac{i\pi}{a}\right)^2 + \left(\frac{j\pi}{b}\right)^2\right\}^2 + \frac{k}{D}\right]} \qquad (9)$$

For the case of zero initial displacement and velocity the solution of Equation 8 may be written as :

$$T_{ij} = \frac{g_{ij}}{mq_{ij}}\int_0^t F(\tau)e^{-r(t-\tau)}\,\mathrm{Sin}\,q_{ij}(t-\tau)\,d\tau \qquad (10)$$

in which q_{ij} is the damped natural circular frequency given by

$$q_{ij} = \sqrt{p^2_{ij} - r^2} \qquad (11)$$

the "Critical damping" c_{cr} for the system can be obetained by setting $q^2_{ij}=0$ thus—

$$c_{cr}(i, j) = 2\sqrt{km}\sqrt{\left[1 + \frac{D}{k}\left\{\left(\frac{i\pi}{a}\right)^2 + \left(\frac{j\pi}{b}\right)^2\right\}^2\right]} \qquad (12)$$

It may be observed that the factor inside the bracket shows effect of the flexural rigidity of the plate and the mode shapes (value of i and j) or the value of the critical damping for the $(i, j)^{th}$ mode.

For any given loading the solution can be obtained from Equation 3 by use of Equation 6 and 10.

1. **Triangular Pulse Loading**—In this case $P(x,y,t) = P\left(1 - \frac{t}{t_1}\right)$ is constant over the entire place and taking $r=0$ (no damping), for a square plate Equation 6 yields—

$$g_{ij} = \frac{16P}{\pi^2\, ij}$$

Equation 10 yields—

$$T_{ij} = \frac{16P}{\pi^2 m_{ij} p^2_{ij}}\int_0^t \left(1 - \frac{\tau}{t_1}\right)\mathrm{Sin}\,p_{ij}(t-\tau)\,d\tau$$

$$= \frac{16P}{\pi^2 m_{ij}\, p^2_{ij}}\left(1 - \frac{t}{t_1} - \mathrm{Cos}\,p_{ij}t + \frac{\mathrm{Sin}\,p_{ij}\,t}{t_1\,p_{ij}}\right)$$

Finally substituting into Equation 3 one has

$$w(x,y,t) = \frac{16P}{\pi^2 m}\sum_{i=1}^{\infty}\sum_{J=1}^{\infty}\frac{i}{ij\,p^2_{ij}}\left[1 - \frac{t}{t_1} + \frac{\mathrm{Sin}\,p_{ij}\,t}{t_1\,p_{ij}\,t} - \mathrm{Cos}\,p_{ij}\,t\right]\mathrm{Sin}\frac{i\pi x}{a}\;\mathrm{Sin}\frac{j\pi y}{b}$$

for $t < t_1$ (13)

2. **Partial loading**—The loading is the same as preceding one except that it is applied over an area uxv whose centre is located at (ξ, η). In this case Equation 6 yields—

$$g_{ij} = \frac{16P}{\pi^2 ij\ uv} \text{Sin} \frac{i\pi x}{a} . \text{Sin} \frac{j\pi n}{b} \text{Sin} \frac{i\pi u}{2a} \text{Sin} \frac{j\pi v}{2b}$$

When the load is concentrated, i.e. u, v→O and P. u. v,→F (a constant), the above equation yields.

$$g_{ij} = \frac{4F}{ab} \text{Sin} \frac{i\pi\xi}{a} \text{Sin} \frac{j\pi\eta}{a} \tag{15}$$

complete solution can be written as before

3. **Rectangular Pulse**—The load P is applied over the plate. In this case the effect of the foundation damping is also included. From Equations 6 and 10 one obtains respectively—

$$g_{ij} = \frac{16P}{\pi^2{}_{ij}} \qquad [16\ (a)]$$

$$T_{ij} = \frac{16P}{m\pi^2 ij\ q_{ij}} \int_0^t e^{-r(t-\tau)} \text{Sin}\ q_{ij}\ (t-\tau)\ d\tau$$

$$= \frac{16P}{\pi^2 ijm} \frac{1}{r^2+q^2_{ij}} \left\{1- \frac{e^{-tr}}{q_{ij}} (\ r \text{ Sin } q_{ij}\ t+q_{ij} \text{ Cos } q_{ij}\ t\ )\right\} \qquad [16\ (b)]$$

The complete solution is given by Equation 3 as :

$$w\ (x,\ y,\ t) = \frac{16P}{m\pi^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ij(r^2+q^2_{ij})} \left[1-e^{-rt} (\frac{r}{q_{ij}} \text{ Sin } q_{ij}\ t+\text{Cos } q_{ij}t)\right]$$

$$\text{Sin} \frac{i\pi x}{a} \text{ Sin} \frac{j\pi y}{b} \text{ For all } t > 0 \text{ and } c_{ij} < c_{cr}\ (i,\ j) \qquad [16\ (c)$$

NUMERICAL EXAMPLE

As a numerical example, a simply supported concrete slab 12'×12'×1' ($E=2\times10^6$ psi) resting over firm soil ($k=614.4$ lbs/in³ and $c=0$) is considered. The first fundamental period ($i=1$, $j=1$) for such a plate is found to be .0093 sec., using Eq. 9 and similarly the value of critical damping for the first mode is found to be c_{cr} (i, j) = 3.51 lbs/in³/sec. It is of interest to note here, that the first fundamental period of the plate corresponding to the first mode ($i=1, j=1$) for $k=0$, $c=0$ is 1.405 times larger i.e. it is .01335 sec. Hence the effect of foundation stiffness is to reduce the period which follows also from mass-spring analogy.

The centre point response is shown in Fig. 3, for the case of a rectangular pulse loading of magnitude 10 psi acting uniformly all over the plate ($t_1=0$) and for $k=614.4$ lbs/in³ and $c=0$. In the same figure is shown the response curve for same 'k' value but $c=3.50$ (slightly less than the critical damping for the first mode). It may be observed that the influence of damping in reducing the magnitude of centre point response is quite pronounced. It may

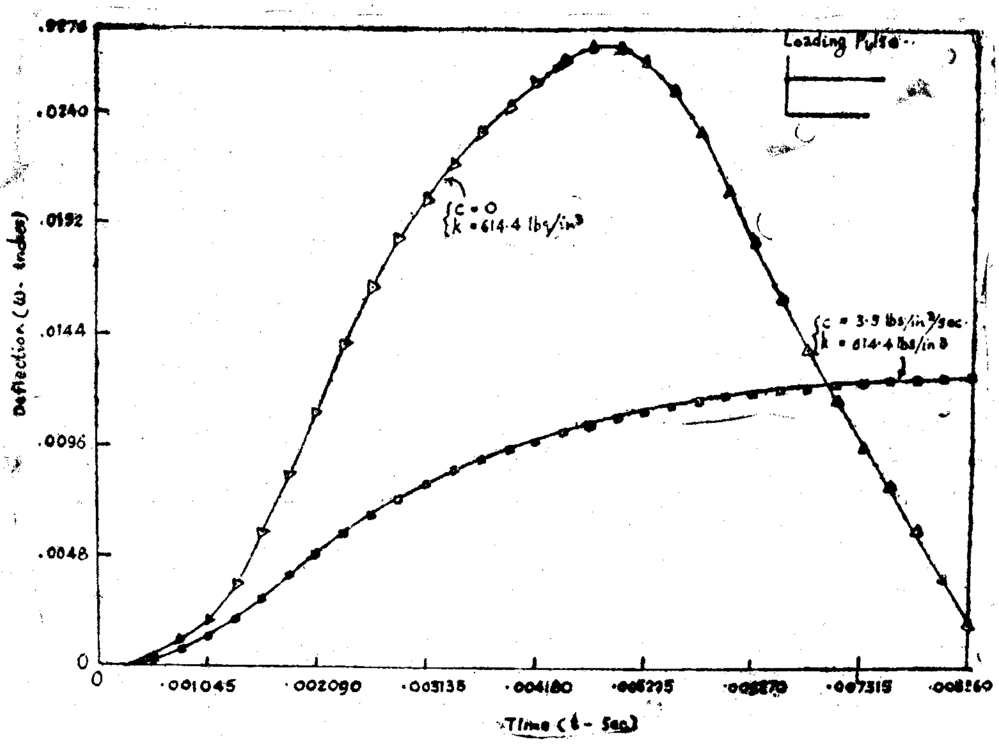


Figure 3 Centre Point Deflection Response

further be noted that for value of $c=3.5$ lbs/in^3/sec, the dynamic response approaches the static response (centre point static deflection=.012971 in.). This would point to the possibility of obtaining static solutions by use of dynamic analysis ("pseudo dynamic approach").

CONCLUSIONS

Navier type solution is convenient and straight forward for solving dynamic response problems of simply supported rectangular plates. However if the boundary conditions are other than those described above this solution will not be a suitable one as is true for the static analysis also Here the vertical reaction of soil has been taken proportional to displacement at that point only which is only an approximation. To consider complex behaviour of soil numerical methods may be used.

ACKNOWLEDGEMENT

This paper is based on the part of author's thesis submitted to Michigan State University, East Lansing Mich. U.S.A. in partial fulfilment of the requirement for the Ph. D degree under guidance of Dr. R.K. Wen, Associate Professor, Department of Civil Engineering.

APPENDIX

NOTATIONS

a=length of the longer side of the plate;

b=length of the shorter side of the plate;

c=foundation damping coefficient;

c_{cr} (i, j)=critical damping for the mode (i,j);

$D=Eh^3/12\ (1-\nu^2)$, flexural rigidity of the plate;

E=modulus of elasticity of the plate material;

F(t)=time dependent part of the forcing function P (x,y,t);

G(x,y)=space function part of the forcing function P (x,y,t);

g_{ij}=Fourier coefficient for G (x,y);

h=plate thickness;

i,j=variable subscripts to denote number of terms in the infinite series;

k=foundation stiffness constant;

m=mass per unit area of plate;

P=magnitude of forcing function;

P(x,y,t)=forcing function;

p_{ij}=natural circular frequency (undamped) of the (i, j)th mode of the plate;

$q_{ij}=\sqrt{p^2_{ij}-r^2}$, damped natural circular frequency corresponding to the (i, j) mode;

q (x,y)=static loading function acting over the plate;

$r=\frac{c}{2m}$, viscous damping parameter;

$S_{ij}=\text{Sin}\ \frac{i\pi x}{a}\ \text{Sin}\ \frac{j\pi y}{b}$ space function;

T_{ij}=time function;

t=time;

t_1=duration of loading pulse;

u=width along x coordinate direction of the partially loaded area;

v=width along y coordinate direction of the partially loaded area;

w=deflection of plate;

x=space coordinate;

y=space coordinate;

Δ^4=biharmonic operator;

ξ=x coordinate of the centre of loaded area;

η=y coordinate of the centre of loaded area; and

ν=Poisson's ratio.

REFERENCES

Sharma, P.C. (1964), "Dynamic Response of Plates on Elastic Foundation", Ph. D. Thesis, Michigan State University, East Lansing, Mich., U.S.A., pp. 74-78.

Timoshenko, S.P. and D.H Young, (1955) "Vibration Problems in Engineering" Third Edition, D. Von Nostrand Company, Inc., New York, pp. 104-109

Timoshenko, S.P. and S. Woinowsky-Krieger, (1959), "Theory of Plates and Shells", Second Edition, McGraw-Hill Book Co., Inc. New York., pp. 79-82