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MATRIX ANALYSIS OF BEAMS ON ELASTIC FOUNDATION SUBJECTED TO STATIC AND DYNAMIC LOADS

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INTRODUCTION

The theory of beams on elastic foundation is commonly used in soil-structure interaction and occupies a very important place in soil mechanics. The methods for the solution due to Winkler's assumption were used (Winkler, 1867) for more than one century. Winkler's hypothesis states that the reactions of the foundation are proportional at every Point to the deflection of the beam at that point. Until recently, a large number of studies and many numerical methods have been proposed by many researchers (Iyenger, 1965 and Malter, 1960) to solve this problem using Winkler's hypothesis. Based on the theory of Vlasov's general variational method (Vlasov et al, 1966), a computer program was developed by Harr (1969) for analysing beams on elastic foundation. Shih-Fang Chen (1972) developed stiffness matrix for beams on elastic foundation using virtual work principle based on Vlasov's theory. In the present investigation authors have used Winkler's hypothesis and the value of '*K*' (Elastic modulus) used was obtained by a number of soil tests around Pilani.

The authors chose to solve a simply supported beam resting on elastic foundation and subjected to static and time varying loads uniformly distributed over the entire length. The analytical solution for the two cases was obtained by solving the equation of motion. The numerical solution was obtained using matrix method, which requires the determination of stiffness, mass and loading matrices. The two results for deflection are compared. The dynamic case is represented graphically and the static case in a tabular form.

PROBLEM FORMULATION

The equation of motion for transverse vibration of a uniform beam of length L resting on elastic foundation and subjected to a general forcing function is given as

$$EI\frac{\partial^4 Y(X,t)}{\partial X^4}+\rho A\frac{\partial^2 Y(X,t)}{\partial t^2}+KY(X,t)=q(X,t), \qquad \ldots(1)$$

where

$\rho=r/g$

r=unit weight of the beam material

g=acceleration due to gravity

EI=flexural stiffness of the beam

A=cross sectional area of the beam element

K=modulus of subgrade reaction

$Y(X,t)$=deflection of the beam at some point X and any time t.

$q(X,t)$=forcing function.

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DYNAMIC CASE (ANALYTICAL SOLUTION)

Assume a series solution for the analytical value of the deflection $Y(X, t)$ of the form

$$Y(X,t)=\sum_{i=1}^{\infty} \phi_i(t) \sin \frac{i\pi X}{L}, \qquad \ldots(2)$$

and setting the value of $Y(X, t)$ and its derivatives from eqn. (2) into eqn. (1) after taking $q(X,t)=q(t)$ (because load is uniformly distributed over the entire length), yields:

$$EI\Sigma \phi_i(t) \frac{i^4\pi^4}{L^4} \sin \frac{i\pi X}{L}+\rho A \Sigma \ddot{\phi}_i(t) \sin \frac{i\pi X}{L}+K\Sigma \phi_i(t) \sin \frac{i\pi X}{L}=q(t) \qquad \ldots(3)$$

Multiplying this equation by $\sin \frac{i\pi X}{L}$ and integrating with respect to X between the limits 0 to L, one obtains:

$$\left(EI\phi_i(t) \frac{i^4\pi^4}{L^4}+\rho A\ddot{\phi}_i(t)+K\phi_i(t)\right)\frac{L}{2}=\frac{2q(t)L}{i\pi}, \quad i=1,3,5 \ldots \qquad \ldots(4)$$

or

$$\ddot{\phi}_i(t)+\phi_i(t)p_i^2=\frac{4q(t)}{i\pi\rho A} \qquad \ldots(5)$$

where

$$p_i^2=\left(\frac{EI\pi^4 i^4+KL^4}{\rho AL^4}\right) \qquad \ldots(6)$$

The general solution of equation (5) can be written as

$$\phi_i(t)=A_i \cos p_i t+B_i \sin p_i t+\frac{4}{i\pi\rho Ap_i}\int_0^t q(T) \sin p_i(t-T)\,dT. \qquad \ldots(7)$$

With the initial conditions

$$Y(X,0)=0, \text{ i.e. } \phi_i(0)=0$$
$$\dot{Y}(X,0)=0, \text{ i.e. } \dot{\phi}_i(0)=0 \qquad \ldots(8)$$

Making use of initial conditions from equation (8) into equation (7), the arbitrary constants A_i and B_i can be evaluated.

Thus $$A_i=B_i=0 \qquad \ldots(9)$$

Therefore;

$$\phi_i(t)=\frac{4}{i\pi\rho Ap_i}\int_0^t q(T) \sin p_i(t-T)\,dT \qquad \ldots(10)$$

The three time distributions of the uniformly distributed load over the entire length of the beam considered are given in case I, II and III and the corresponding values of $\phi_i(t)$ are also listed for the analytical solution of the problem.

Case I:

$$q(t)=q_0 t/t_0 \qquad \text{[Fig. 1 (i)]}$$

$$\phi_i(t)=\frac{4q_0}{\rho A\pi t_0 i p_i^2}\left(t-\frac{\sin p_i t}{p_i}\right), \quad i=1, 3, 5 \ldots \text{ for } t \leqslant t_0$$

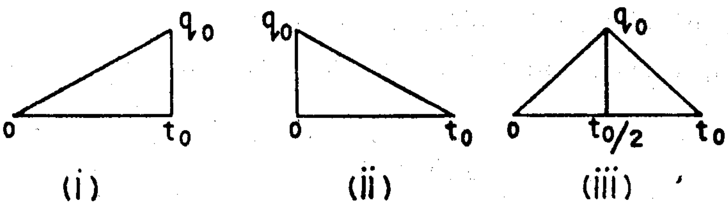


Fig. 1 Time Distribution of Load.

and

$$\phi_i(t)=\frac{4q_0}{\rho\pi A t_0 i p_i^2}\Big(t_0\cos p_i(t-t_0)+\frac{\sin p_i(t-t_0)}{p_i}-\frac{\sin p_i t}{p_i}\quad \text{for } t\geqslant t_0 \qquad \ldots(11)$$

$$Y(X,t)=\sum_{i=1}^{\infty}\phi_i(t)\sin\frac{i\pi X}{L} \qquad \ldots(12)$$

Case II :

$$q(t)=q_0\left(1-\frac{t}{t_0}\right)\ \text{[Fig. 1(ii)]}$$

$$\phi_i=\frac{4q_0}{\rho\pi A i p_i^2}\left(1-\cos p_i t-\frac{t}{t_0}+\frac{\sin p_i t}{p_i t_0}\right)\ \text{for } t\leqslant t_0$$

and

$$\phi_i=\frac{4q_0}{\rho A\pi i p_i^2}\left(-\cos p_i t-\frac{\sin p_i(t-t_0)}{p_i t_0}+\frac{\sin p_i t}{p_i t_0}\right)\ \text{for } t\geqslant t_0 \qquad \ldots(13)$$

Case III :

(Combination of Case I and II) [Fig. 1(iii)]

when $t\leqslant t_0/2$

$$\phi_i=\frac{8q_0}{i\rho A\pi t_0 p_i^2}\left(t-\frac{\sin p_i t}{p_i}\right), \qquad \ldots(14)$$

and $\frac{t_0}{2}\leqslant t\leqslant t_0$

$$\phi_i=\frac{8q_0}{i\rho A\pi t_0 p_i^2}\left(t_0-t+\frac{2\sin p_i(t-t_0/2)}{p_i}-\frac{\sin p_i t}{p_i}\right) \qquad \ldots(15)$$

$$\phi_i=\frac{8q_0}{i\rho A\pi t_0 p_i^3}\,(2\sin p_i(t-t_0/2-\sin p_i(t-t_0-\sin p_i t)\ \text{for } t\geqslant t_0 \qquad \ldots(15a)$$

MATRIX ANALYSIS

This method consists in the derivation of stiffness, load and mass matrices for the beam elements. (The beam is divided in certain number of parts and each part is termed as beam element)

For generating stiffness matrix, the rate of loading on the beam should be set to zero and thus, the equation of the beam resting on elastic foundation in its revised form is

$$EI\frac{d^4Y}{dX^2}+KY=0 \qquad \ldots(16)$$

The general solution of this equation is given as

$$Y = e^{ZX}(C_1 \cos ZX + C_2 \sin ZX) + e^{-ZX}(C_3 \cos ZX + C_4 \sin ZX) \quad \ldots(17)$$

where

$$Z = 4\sqrt{K/4EI} \quad \ldots(18)$$

The simple beam element is first fixed at the two ends and then unit displacements are produced in 1, 2, 3 and 4 directions, as given in Fig. 2. Thus there are four cases of boundary conditions to be considered in getting the stiffness matrix:

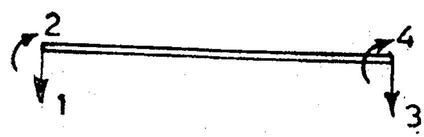


Fig. 2 End Displacements of a Beam Element.

Case A :

$$\text{at} \quad X=0; \quad Y=1, \quad \frac{dX}{dX}=0$$

$$\text{at} \quad X=l; \quad Y=0, \quad \frac{dY}{dX}=0 \quad \ldots(19)$$

Case B :

$$\text{at} \quad X=0; \quad Y=0, \quad \frac{dY}{dX}=1$$

$$\text{at} \quad X=l; \quad Y=0, \quad \frac{dY}{dX}=0 \quad \ldots(20)$$

Case C :

$$\text{at} \quad X=0, \quad Y=0, \quad \frac{dY}{dX}=0$$

$$\text{at} \quad X=l, \quad Y=1, \quad \frac{dY}{dX}=0 \quad \ldots(21)$$

Case D :

$$\text{at} \quad X=0, \quad Y=0, \quad \frac{dY}{dX}=0$$

$$\text{at} \quad X=l, \quad Y=0, \quad \frac{dY}{dX}=1 \quad \ldots(22)$$

where l is the length of beam element.

Renamnig the arbitrary constants C_1, C_2, C_3 and C_4 of equation (17) as C_{11}, $C_{12}, \ldots C_{14}$, C_{21}, $C_{22} \ldots C_{24}$, C_{31}, $C_{32} \ldots C_{34}$, C_{41}, $C_{43} \ldots C_{44}$ for the four different cases of boundary conditions discussed, one can obtain these constants by setting the boundary conditions from eqns. (19), (20), (21) and (22) into eqn. (17). Thus the arbitrary constants are as follows:

Case A :

$$C_{11}=(A_3^2+A_4^2-A_1A_3-A_1A_4-A_2A_3-3A_2A_4)/D \qquad \ldots(23)$$

$$C_{12}=-(A_3^2+A_4^2-3A_1A_4-A_1A_3+A_2A_3+A_2A_4)/D \qquad \ldots(24)$$

$$C_{13}=(A_1^2+A_2^2-A_1A_3+A_1A_4+A_2A_3-3A_2A_4)/D \qquad \ldots(25)$$

$$C_{14}=(A_1^2+A_2^2-A_1A_3-A_1A_4+A_2A_4+3A_2A_3)/D \qquad \ldots(26)$$

where

$$D=A_1^2+A_2^2+A_3^2+A_4^2-2A_1A_3-6A_2A_4 \qquad \ldots(27)$$

and

$$A_1=e^{Zl}\cos Zl,\ A_2=e^{Zl}\sin Zl$$

$$A_3=e^{-Zl}\cos Zl \text{ and } A_4=e^{-Zl}\sin Zl \qquad \ldots(28)$$

Case B :

$$C_{21}=-\frac{1}{ZD}(A_1A_4+2A_2A_4-A_2A_3) \qquad \ldots(29)$$

$$C_{22}=\frac{1}{ZD}(A_3^2+A_4^2-A_1A_3+2A_1A_4-A_2A_4) \qquad \ldots(30)$$

$$C_{23}=-C_{21} \qquad \ldots(31)$$

$$C_{24}=\frac{1}{ZD}(A_1^2+A_2^2-A_1A_3-2A_2A_3-A_2A_4) \qquad \ldots(32)$$

Case C :

$$C_{31}=-\frac{1}{D}(A_3-A_4-A_1-A_2) \qquad \ldots(33)$$

$$C_{32}=(-A_1+A_2+A_3-3A_4)/D \qquad \ldots(34)$$

$$C_{33}=(A_3-A_4-A_1-A_2)/D \qquad \ldots(35)$$

$$C_{34}=-(A_1+3A_2-A_3-A_4)/D \qquad \ldots(36)$$

Case D :

$$C_{41}=-\frac{1}{ZD}(A_2-A_4) \qquad \ldots(37)$$

$$C_{42}=\frac{1}{ZD}(A_1-A_3-2A_4) \qquad \ldots(38)$$

$$C_{43}=-C_{41} \qquad \ldots(39)$$

$$C_{44}=\frac{1}{ZD}(-A_1+2A_2+A_3) \qquad \ldots(40)$$

Defining these constants

$$\left.\begin{aligned} B_1&=A_1-A_2\\ B_2&=A_1+A_2\\ B_3&=-(A_3+A_4)\\ B_4&=A_3-A_4 \end{aligned}\right\} \qquad \ldots(41)$$

and writing down the general expression for shear as $EI\dfrac{d^3Y}{dX^3}$ and for moment as $EI\dfrac{d^2Y}{dX^2}$. One can write down the expression for shear and moment at the two ends of the beam element under consideration.

Therefore:

$$\text{at} \quad X=0,\ \text{moment}=M_{i1}=2\,(C_{12}-C_{14})\,Z^2EI$$

$$\text{shear}=Q_{i_1}=2\,(-C_{i1}+C_{i2}+C_{i3}+C_{i4})\,Z^3EI \qquad \ldots(42)$$

$$\text{at} \quad X=l,\ M_{i_2}=2\,(-A_2C_{i1}+A_1C_{i2}-A_3C_{i4}+A_4C_{i3})\,Z^2EI$$

$$Q_{i2}=2\,(+B_1C_{i2}-B_2C_{i1}-B_3C_{i4}+B_4C_{i3})\,Z^3EI \qquad \ldots(43)$$

Thus one can generate the stiffness matrix which will be a 4×4 matrix. The first row for, example is given as

$$2Z^2EI\,[Z(-C_{i_1}+C_{i2}+C_{i3}+C_{i4}),\ -(C_{i2}-C_{i4}),$$
$$Z\,(B_2C_{i_1}-B_1C_{i2}-B_4C_{i3}+B_3C_{i4}),$$
$$(-A_2C_{i1}+A_1C_{i3}+A_4C_{i3}-A_3C_{i4})] \qquad \ldots(44)$$

Thus the complete stiffness matrix (S) can be written as follows:

$$S_{11}=2Z\,(e^{2Zl}+2\sin 2Zl-e^{-2Zl})/U=S_{33}$$
$$S_{12}=(e^{2Zl}+e^{-2Zl}-2\cos 2Zl)/U=S_{21}=-S_{34}=-S_{43}$$
$$S_{13}=-4Z\,[e^{Zl}(\cos Zl+\sin Zl)-e^{-Zl}(\cos Zl-\sin Zl)]/U=S_{31} \qquad \ldots(44a)$$
$$S_{14}=4\,(e^{Zl}-e^{-Zl})\sin Zl/U=S_{41}=-S_{23}=-S_{32}$$
$$S_{22}=(e^{2Zl}-e^{-2Zl}-2\sin 2Zl)/ZU=S_{44}$$
$$S_{24}=-2\,[e^{Zl}(\cos Zl-\sin Zl)-e^{-Zl}(\sin Zl+\cos Zl)]/ZU=S_{42}$$

when S_{ij} is the element of the stiffness matrix $S(4\times4)$ corresponding to i^{th} row and j^{th} column and this is symmetrical and

$$U=e^{2Zl}+e^{-2Zl}-2-4\sin^2 Zl \qquad \ldots(44b)$$

DERIVATION OF LOAD MATRIX

Writing down eqn. (16) for the forcing function q, yields;

$$EI\,\frac{d^4Y}{dX^4}+KY=q \qquad \ldots(45)$$

The general solution of this equation is

$$Y=e^{ZX}(C_1\cos ZX+C_2\sin 2ZX)+e^{-ZX}(C_3\cos ZX$$
$$+C_4\sin ZX)+q/K \qquad \ldots(46)$$

together with boundary conditions

$$Y(0)=\frac{dY}{dX}(0)=0$$

$$Y(l)=\frac{dY}{dX}(l)=0 \qquad \ldots(47)$$

Setting these boundry condition in eqn. (46) one obtains

$$C_1=-\frac{q}{KD}(A_3^2+A_4^2-A_1A_3-A_1A_4-A_2A_3-3A_2A_4-A_3+A_4+A_1+A_2) \quad \ldots(48)$$

$$C_2=\frac{q}{KD}(A_3^2+A_4^2-A_1A_3-3A_1A_4+A_2A_3+A_2A_4+A_1-A_2-A_3+3A_4) \quad \ldots(49)$$

$$C_3=-\frac{q}{KD}(A_1^2+A_2^2-A_1A_3+A_1A_4+A_2A_3-3A_2A_4+A_3-A_4-A_1-A_2) \quad \ldots(50)$$

and

$$C_4=-\frac{q}{KD}(A_1^2+A_2^2-A_1A_3-A_1A_4+3A_2A_3+A_2A_4-A_1-3A_2+A_3+A_4) \quad \ldots(51)$$

where the constants have been defind earlier.

Thus the loading matrix can be written as

$$2Z^2EI\begin{vmatrix} -Z(-C_1+C_2+C_3+C_4) \\ (C_2-C_4) \\ -Z(B_2C_1-B_1C_2-E_4C_1+B_3C_4) \\ (A_2C_1-A_1C_2-A_4C_3+A_3C_4) \end{vmatrix} \quad \ldots(52)$$

DERIVATION OF MASS MATRIX

Defining the elemental mass matrix for an element as follows (Archer, 1969) :

$$m_{ij}=\rho A\int_o^l Y_i\,Y_j\,dX \quad \ldots(53)$$

where Y_i and Y_j are the displacements of the i^{th} and j^{th} element.

Substituting the values of element displacement already obtained, the mass matrix m general for an element can be given as follows, where m_{ij} is one element (i^{th} row j^{th} col) of the mass matrix:

$$m_{ij}=\frac{1}{8Z}(-3C_{i1}C_{j1}-C_{i2}C_{j2}+3C_{i3}C_{j3}+C_{i4}C_{j4}+C_{i1}C_{j2}$$

$$+C_{i3}C_{j1}+C_{i3}C_{j4}+C_{i4}C_{j3}+2C_{i3}C_{j2}+2C_{i4}C_{j1}$$

$$+2C_{i2}C_{j3}+2C_{i1}C_{j4})+\frac{l}{2}(C_{i3}C_{j1}+C_{i1}C_{j3}$$

$$+C_{i4}C_{j2}+C_{i2}C_{j4})+\frac{1}{4Z}(C_{i3}C_{j1}+C_{i1}C_{j3}$$

$$-C_{i4}C_{j2}-C_{i2}C_{j4})\sin(2Zl)-\frac{1}{4Z}(C_{i3}C_{j2}$$

$$+C_{i4}C_{j1}+C_{i2}C_{j3}+C_{i1}C_{j4})\cos(2Zl)$$

$$+\frac{e^{2Zl}}{8Z}[2C_{i1}C_{j1}+2C_{i1}C_{j2}+(C_{i1}C_{j1}-C_{i2}C_{j2})$$

$$(\sin 2Zl+\cos 2Zl)+(C_{i_1}C_{j_2}+C_{i_2}C_{j_1})$$
$$(\sin 2Zl-\cos 2Zl)]$$
$$+\frac{e^{-2Zl}}{8Z}[-2C_{i_3}C_{j_3}-2C_{i_4}C_{j_4}+(C_{i_3}C_{j_3}-C_{i_3}C_{j_4})$$
$$(\sin 2Zl-\cos 2Zl)-(C_{i_3}C_{j_4}+C_{i_4}C_{j_3})$$
$$(\cos 2Zl+\sin 2Zl)] \qquad \ldots(54)$$

The general equation from where the dynamic displacements are extracted is given below :

$$(m_{ij})(\ddot{Y}_j)+(S_{ij})(Y_j)=(q_i), \qquad \ldots(55)$$

where

m_{ij}=mass matrix

q_i =Loading matrix

S_{ij}=stiffness matrix

STATIC CASE (ANALYTICAL SOLUTION)

The general equation for the static case and its general solution is same as equation (45) and (46) respectively.

The boundary conditions in this case are follows:

$$Y(0)=\frac{d^2Y(0)}{dX^2}=0$$

$$Y(L)=\frac{d^2Y(L)}{dX^2}=0 \qquad \ldots(56)$$

Setting these boundary conditions in equation (46), one gets the value of the arbitrary constants.

$$C_1=\frac{q}{KD_1}(a_3^2+a_4^2-a_1a_3+a_2a_4+a_1-a_3) \qquad \ldots(57)$$

$$C_2=\frac{q}{KD_1}(a_1a_4-a_2a_3+a_2+a_4), \qquad \ldots(58)$$

$$C_3=\frac{q}{KD_1}(a_1^2+a_2^2-a_1a_3+a_2a_4-a_1+a_3) \qquad \ldots(59)$$

and

$$C_4=C_2 \qquad \ldots(60)$$

where

$$D_1=-(a_1^2+a_2^2+a_3^2+a_4^2-2a_1a_3+2a_2a_4)$$
$$a_1=e^{ZL}\cos ZL,\; a_2=e^{ZL}\sin ZL$$
$$a_3=e^{-ZL}\cos ZL,\; a_4=e^{-ZL}\sin ZL \qquad \ldots(61)$$

STATIC CASE (MATRIX ANALYSIS)

The governing equation of the static deflection is given as

$$(S_{ij})(Y_j)=(Q_i) \qquad \ldots(62)$$

or

$$(Y_j)=(S_{ij})^{-1}(Q_i) \qquad \ldots(63)$$

Where S_{ij} and Q_i are the stiffness and loading matrices already defined and Y_j are the elemental displacements of the beam.

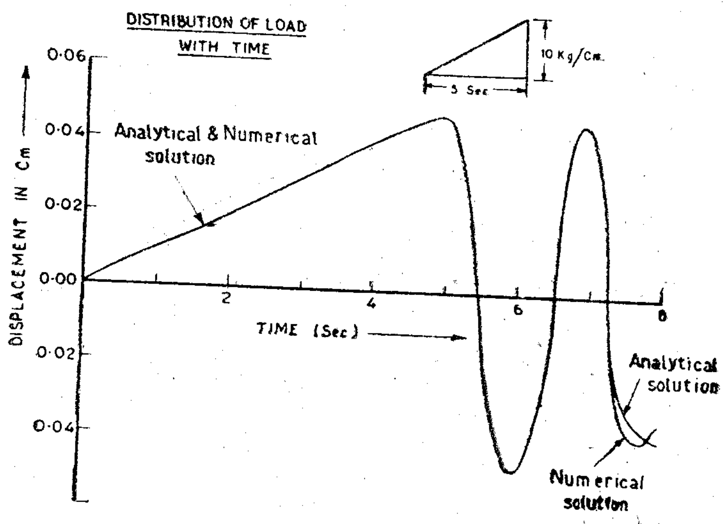


Fig. 3 Displacement Curve at Mid Span.

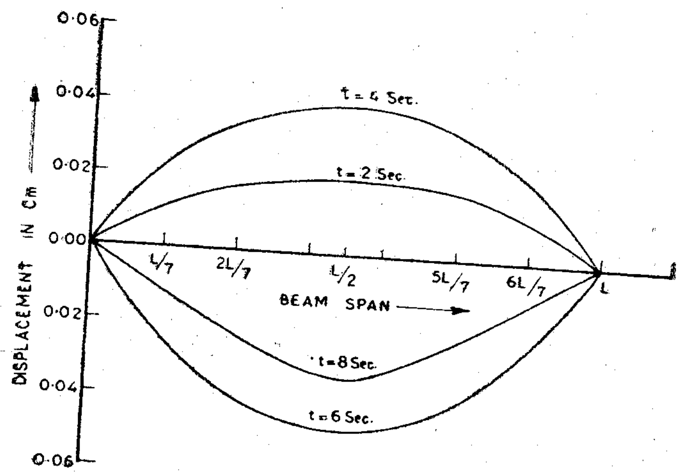


Fig. 4 Beam Displacement at Different Periods.

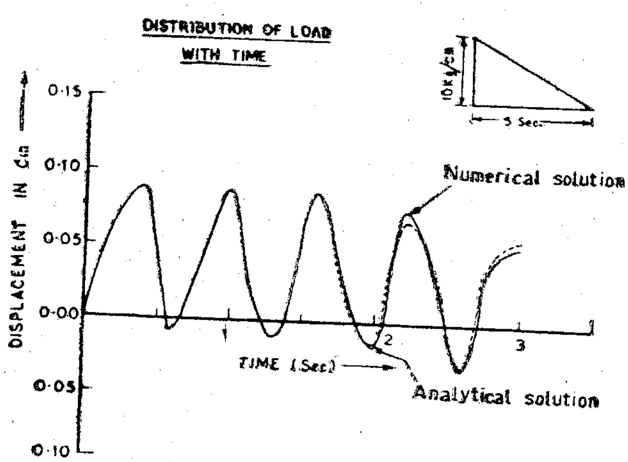


Fig. 5 Time Displacement Curve at Mid-Span.

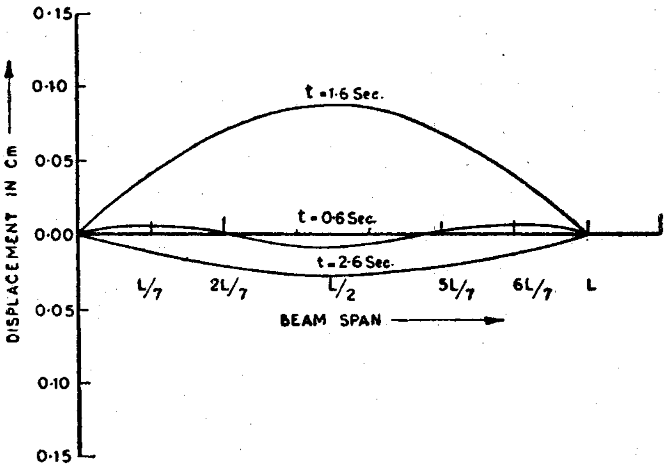
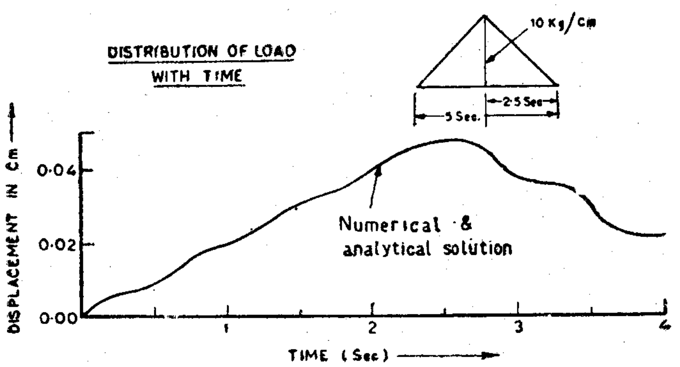


Fig. 6 Beam Displacement at Different Periods.



Fig, 7 Time Displacement Curve at Mid-Span.

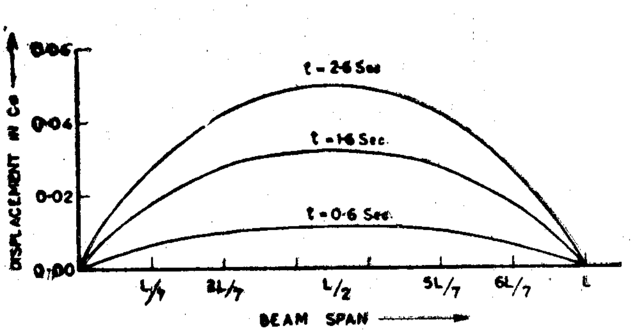


Fig. 8 Beam Displacement at Different Periods.

RESULTS AND DISCUSSION

Thus, for three different time distributions of the dynamic load, the mid span displacements are calculated for different times analytically and numerically and they are plotted in Figs. (1), (3) and (5). As is evident from these figures, there is not much difference in the values of deflection obtained by the numerical method as compared to analytical method, thus proving the accuracy of the matrix method even though, the element size* is not very small and could be made smaller for a larger computer. Beam displacements are also plotted for different times along the length of the beam for three distributions of dynamic load, as shown in Figs. (2), (4) and (6). Except in Fig. (4) it only shows the first vibrational mode. For static loading case, the two deflection values as given in Table—A are exactly same for both analytical method and matrix method.

TABLE A

STATIC LOADING

S. No.	*Span Lengths*	*Analytical Deflections In cms.*	*Deflections by Numerical Method In cms.*
1	0	0	0
2	L/7	0.026985	0.026985
3	2L/7	0.042689	0.042689
4	3L/7	0.048700	0.048700
5	4L/7	0.048700	0.048700
6	5L/7	0.042689	0.042689
7	6L/7	0.026985	0.026985
8	L	0	0

CONCLUSIONS

The simple beam subjected to static and time varying loads, resting on Winkler foundation is completely analyzed and the deflections compared to exact analytical solution. In the case of dynamic load for the three time distribution of the uniform load, the analytical and numerical solutions are very close as is evident from Figs. (1), (3)

* See Appendix.

and (5). This proves the degree of accuracy of the numerical approach. The three vibrational modes are represented in Figs. (2), (4) and (6). The vibrational mode of third case Fig. (6) is not a well defined one because the period of application of the load considered was only 5 secs. In this case we need a larger period to get one cycle of vibration.

The static case deflections by analytical and numerical solutions are exactly same. Since the numerical solution was developed from the analytical solution, the computational errors are completely eliminated.

The next step should be to solve more complex problems of platies and shells using a nonlinear foundation model.

APPENDIX

The beam considered has following material properties.

Span of the beam $L=7.0$ metres

Element size l considered=50 cms.

Cross Sectional area of beam element=65 cm^2 element.

$I=13200$ cm^4

$E=2\times10^6$ kg/m^2

Load=10 kg/cm

$q_0=10$ kg

Duration of dynamic load=5 secs.

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