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SH-WAVES FROM A TORSIONAL SOURCE IN AN ANISOTROPIC HETEROGENEOUS HALF-SPACE[1]

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Introduction

Source problems in homogeneous and inhomogeneous isotropic media have been discussed by many investigators since the classical paper by Lamb (1904). The displacements have been studied by considering different types of fixed or moving sources. A brief account of these investigations has been given by Ewing, Jardetzky and Press (1957). Mitra (1958), De (1969), Vlaar (1966, a, b) and others have studied the displacements of SH-waves in heterogeneous isotropic media. Mitra (1958) has found normal modes and surface displacements for SH-waves from a prescribed surface traction in a half-space. The variations of rigidity and density in the half-space are assumed to very as exp (αz) and exp (βz) respectively, z being the depth coordinate. De has considered the medium in which the density and rigidity both vary as exp ($2 \alpha z$). Vlaar (1966) has discussed the displacements due to SH-waves from the point source in a (i) layer and (ii) half-space in which rigidity and density are assumed to be piece-wise continuous functions of depth. He has studied the normal modes as functions of the source depth and frequency by using Strum Liouville's Theory of eigenfunction expansion. Sidhu (1971) and Bhattacharya (1973) have discussed SH-waves from torsional sources in a semi-infinite elastic medium. Bhattacharya (1973) has obtained displacements in a homogeneous isotropic half-space due to torsional disturbance produced by an expanding source. He has derived the solutions for two different cases where (i) the velocity of the expansion of the source is less than that of shear-waves in the medium (ii) the velocity of expansion of the source is greater than that of the shear-waves in the medium. He has used integral transform technique and the modified Cagniard method. Sidhu (1971) obtained the formal solution of the displacements field due to SH-waves from burried, time harmonic, torsional sources in a half-space in which μ and ρ vary continuously with depth. He has applied this formal solution to study the SH-waves from a torsional source in a half-space in which μ and ρ vary exponentially with depth.

In the present study, the propagation of SH-waves generated by a time-harmonic torsional source in an inhomogeneous, transversely isotropic half-space has been discussed. The elastic parameters of the material are supposed to be functions of the vertical coordinate. The variations of directional rigidities and density have been assumed in such a way that vertical shear wave velocity varies linearly with depth whereas the horizontal shearwave velocity remains constant. The theory of Hankel-transformation is used and the method of stationary phase has been applied to evaluate the surface displacement at large distances from a surface source.

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Formulation of the Problem

We use cylinderical coordinates system with origin at the free surface and z-axis directed into the half-space which is supposed to be transradially isotropic and inhomogeneous. The inhomogeneity in the medium has been taken to be a function of vertical coordinate only. Let the displacement components in the directions of r, ϕ, z be u, v and w respectivel. Since the material is transradially isotropic, the surfaces of constant isotropy are cylinders with axis along z-axis.

In cylindrical coordinates the strain-energy function, W, is given by (Love, 1927)

$$2W = c_{33} e_{rr}^2 + c_{11}(e_{\phi\phi}^2 + e_{zz}^2) + 2 c_{11} e_{\phi\phi} e_{zz} + 2c_{13}(e_{\phi\phi} + e_{zz}) e_{rr} + 2 c_s e_{\phi z}^2 + 2 c_{44}(e_{r\phi}^2 + e_{zr}^2) \tag{1}$$

where $c_s = (c_{11} - c_{12})/2$.

Stress-displacement relations are given by

$$\begin{aligned}
p_{rr} &= c_{33} u_r + c_{13}\left(\frac{1}{r} v_\phi + \frac{u}{r}\right) + c_{13} w_z \\
p_{\phi\phi} &= c_{13} u_r + c_{11}\left(\frac{1}{r} v_\phi + \frac{u}{r}\right) + c_{12} w_z \\
p_{zz} &= c_{13} u_r + c_{12}\left(\frac{1}{r} v_\phi + \frac{u}{r}\right) + c_{11} w_z \\
p_{r\phi} &= c_{44}\left(\frac{1}{r} u_\phi + v_r - \frac{v}{r}\right) \\
p_{\phi z} &= c_s\left(\frac{1}{r} w_\phi + v_z\right) \\
p_{zr} &= c_{44}(u_z + w_r)
\end{aligned} \tag{2}$$

where $u_r = \frac{\partial u}{\partial r}$ etc.

Equations of motion in cylindrical coordinates with body fores (F_r, F_ϕ, F_z) are given by

$$\begin{aligned}
&\frac{\partial}{\partial r}(p_{rr}) + \frac{1}{r}\frac{\partial}{\partial \phi}(p_{r\phi}) + \frac{\partial}{\partial z}(p_{rz}) + \frac{p_{rr} - p_{\phi\phi}}{r} + F_r = \rho \frac{\partial^2 u}{\partial t^2} \\
&\frac{\partial}{\partial r}(p_{r\phi}) + \frac{1}{r}\frac{\partial}{\partial \phi}(p_{\phi\phi}) + \frac{\partial}{\partial z}(p_{\phi z}) + \frac{2p_{r\phi}}{r} + F_\phi = \rho \\
&\frac{\partial}{\partial r}(p_{rz}) + \frac{1}{r}\frac{\partial}{\partial \phi}(p_{z\phi}) + \frac{\partial}{\partial z}(p_{zz}) + \frac{p_{rz}}{r} + F_z = \rho \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{3}$$

We consider the SH-tpye disturbances due to the torsional source defined by

$$F_r \neq 0, \quad F_z = 0, \quad F_\phi = g(r)\, \delta(z-h)\, e^{ipt} \tag{4}$$

where $\delta(z-h)$ is the Dirac-delta function. The only nonzero component of displacement is v and it is independent of ϕ-coordinate. In this case the stress-strain relations become

$$\begin{aligned}
&p_{rr} = p_{\phi\phi} = p_{zz} = p_{zr} = 0 \\
&p_{r\phi} = c_{44}\left(v_r - \frac{v}{r}\right), \quad p_{\phi z} = c_s v_z
\end{aligned} \tag{5}$$

and the equations of motion reduce to

$$c_{44}\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right]+\frac{\partial}{\partial z}\left[c_s\frac{\partial v}{\partial z}\right]+F_\phi=\rho\frac{\partial^2 v}{\partial t^2} \tag{6}$$

Assuming that v varies as exp (ipt)

we have

$$c_{44}\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right]+\frac{\partial}{\partial z}\left[c_s\frac{\partial v}{\partial z}\right]+F_\phi+\rho p^2 v=0 \tag{7}$$

Applying the Hankel transform defined by

$$v=\int_0^\infty v\,J_1(\xi r)\,r dr,\quad G(\xi)=\int_0^\infty g(r)\,J_1(\xi r)\,r dr \tag{8}$$

to the equation (7) we get

$$\frac{dv}{dz}\left[c_s\frac{dv}{dz}\right]+[\rho p^2-c_{44}\xi^2]\,v=-G(\xi)\,\delta(z-h) \tag{9}$$

Since $(z-h)=0$ for $z \lessgtr h$, therefore equation (9) becomes

$$\frac{d}{dz}\left[c_s\frac{dv}{dz}\right]+(\rho p^2-c_{44}\xi^2)\,v=0 \tag{10}$$

for $z > h$ and $z < h$.

Let the two linearly independent solutions of this equation be v_{11} and v_{12} so that one of them (say) remains bounded as $z\to\infty$.

The most general form of the solution of (10) satisfying the radition conditions, is then

$$v=\begin{cases} v_1=c_1 v_{11}+c_2 v_{12} & \text{for } z<h \\ v_2=c_3 v_{12} & \text{for } z>h \end{cases} \tag{11}$$

where c_1, c_2 and c_3 are certain unknown constants to be determined from the following boundary conditions :

(i) The surface $z = 0$ is stress free, i.e.

$$p_{z\phi}=0 \text{ at } z=0 \tag{12}$$

(i i) As the body force F_ϕ is assumed to operate between the elastic limits, the displacement is continuous everywhere and in particular at $z = h$, i.e.,

$$v_1=v_2 \text{ at } z=h \tag{13}$$

(iii) The solution given by (11) does not appear to take the source effect. This is achieved by integrating (9) within the limits $h-\epsilon$ and $h+\epsilon$ and then taking the limits a $\epsilon\to 0$. This gives

$$c_s(h)\,[v_1'(h)-v_2'(h)]=G(\xi) \tag{14}$$

Substituting the values of v_1 and v_2 from (11) into (12), (13) and (14) we get the following set of three equations

$$c_1 v_{11}'(o) + c_2 v_{12}'(o) = 0$$

$$c_1 v_{11}(h) + c_2 v_{12}(h) = c_3 v_{12}(h)$$

and

$$c_s(h)\,[c_1 v_{11}'(h) + c_2 v_{12}'(h) - c_3 v_{12}'(h)] = G(\xi).$$

Solving these equations for c_1, c_2 and c_3, we get

$$c_1 = -\frac{G(\xi)\, v_{12}(h)}{\Delta}$$

$$c_2 = \frac{G(\xi)\, v_{12}(h)\, v_{11}(0)}{\Delta\, v_{12}(0)} \tag{15}$$

$$c_3 = -\frac{G(\xi)\, v_{11}(h)}{\Delta} + c_2$$

where $\Delta = c_s(h)\,[v_{11}(h)\, v_{12}'(h) - v_{12}(h)\, v_{11}'(h)]$ (16)

Using the Hankel inverse transform we get

$$v = \begin{cases} \int_0^\infty (c_1 v_{11} + c_{12} v_{12} J_1(r\xi)\,\xi\, d\xi & \text{for } z < h \\ \int_0^\infty (c_3 v_{12} J_1(r\xi)\,\xi\, d\xi & \text{for } z > h \end{cases} \tag{17}$$

Applications to Hetrogeneous Medium

We consider the variations of the elastic parameters of the medium in such a way that the vertical shear-wave velocity varies linearly with depth from the surface and the horizontal shear-wave velocity remains constant.

Let the inhomogeneity in the medium be defined by

$$\left.\begin{aligned} c_s &= \bar{c}_s (1 + \alpha z)^3 \\ c_{44} &= \bar{c}_{44} (1 + \alpha z) \\ \rho &= \bar{\rho} (1 + \alpha z) \end{aligned}\right] \tag{18}$$

where $\bar{c}_s$, $\bar{c}_{44}$ and $\bar{\rho}$ are the values of the elastic parameters at the surface and α is a nonzero constant. Substituting the above values in equation (10) we get

$$\frac{d^2v}{dz^2} + \frac{3\alpha}{1+\alpha z}\frac{dv}{dz} - \frac{1}{r_0^2}\left(\xi^2 - \frac{p^2}{v_{v0}^2}\right)\frac{v}{(1+\alpha z)^2} = 0 \tag{19}$$

where

$$\bar{c}_s = \bar{\rho}\, v^2_{H0},\ \bar{c}_{44} = \bar{\rho}\, v^2_{v0},\ r_0 = v_{H0}/v_{v0} \tag{20}$$

Let $1 + \alpha z = Z$, Therefore

$$\frac{d^2v}{dz^2} + \frac{3}{z}\frac{dv}{dz} - \frac{1}{\alpha^2 r_0^2}\left(\xi^2 - \frac{p^2}{v^2_{v0}}\right)\frac{v}{z^2} = 0$$

or

$$z^2\frac{d^2v}{dz^2} + 3z\frac{dv}{dz} - \beta^2 v = 0 \tag{20 a}$$

where $$\beta^2 = \frac{1}{\alpha^2 v_0^2}\left(\xi^2 - \frac{p^2}{v_0^2}\right) \tag{21}$$

Solution of (20 a) is given by

$$v = \begin{cases} c_1 (1+\alpha z)^{[-1+\sqrt{1+\beta^2}]} + c_2 (1+\alpha z)^{-[1+\sqrt{1+\beta^2}]} & \text{for } z < h \\ c_3 (1+\alpha z)^{-[1+\sqrt{1+\beta^2}]} & \text{for } z > h \end{cases} \tag{22}$$

where c_1, c_2 and c_3 are certain unknown constants, from (15) and (22) we get

$$c_1 = \frac{G(\xi)\,(1+\alpha h)^{-[1+\beta_1]}}{2\alpha\,\bar{c}_s\,\beta_1}$$

$$c_2 = \frac{G(\xi)\,(1+\alpha h)^{-[1+\beta_1]}}{2\alpha\,\bar{c}_s\,\beta_1} \times \frac{(-1+\beta_1)}{(1+\beta_1)}$$

$$c_3 = \frac{G(\xi)\,(1+\alpha h)^{-1}}{2\alpha\,\bar{c}_s\,\beta_1\,(1+\beta_1)}\left[(1+\beta_1)(1+\alpha h)^{\beta_1} + (\beta_1-1)(1+\alpha h)^{-\beta_1}\right] \tag{23}$$

where $$\beta_1 = \sqrt{1+\beta^2} \text{ and } \beta_1 \neq -1 \tag{24}$$

Therefore from (17), (22) and (23) we get

$$v = \begin{cases} \dfrac{e^{ipt}}{2\alpha\,\bar{c}_s}\displaystyle\int_0^\infty \frac{G(\xi)\,(1+\alpha h)^{-(1+\beta_1)}}{\beta_1(1+\beta_1)}\left[(1+\beta_1)(1+\alpha z)^{\beta_1-1} + (\beta_1-1)(1+\alpha z)^{-(\beta_1+1)}\right] J_1(r\xi)\,\xi\,d\xi & \text{for } z < h \\ \dfrac{e^{ipt}}{2\alpha\,\bar{c}_s}\displaystyle\int_0^\infty \frac{G(\xi)\,(1+\alpha h)^{-1}}{\beta_1(1+\beta_1)}\left[(1+\beta_1)(1+\alpha h)^{\beta_1} + (\beta_1-1)(1+\alpha h)^{-\beta_1}\right](1+\alpha z)^{-(\beta_1+1)} J_1(r\xi)\,\xi\,d\xi & \text{for } z > h \end{cases} \tag{25}$$

In order to obtain the surface displacement, we put $z = 0$ in the first equation of 25 and get

$$v = \frac{e^{ipt}}{\alpha\,\bar{c}_s}\int_0^\infty \frac{G(\xi)\,(1+\alpha h)^{-(1+\beta_1)}}{1+\beta_1} J_1(r\xi)\,\xi\,d\xi \tag{26}$$

The corresponding case when the source is at the surface we have $h = 0$ and therefore from (25),

$$v = \frac{e^{ipt}}{\alpha\,\bar{c}_s}\int_0^\infty \frac{G(\xi)\,(1+\alpha z)^{-(\beta_1+1)}}{(1+\beta_1)} J_1(r\xi)\,\xi\,d\xi \tag{27}$$

In order to obtain the displacement at a depth h when the source is in the free surface, we put $z = h$ in (27) and get

$$v = \frac{e^{ipt}}{\alpha c_s}\int_0^{\infty} \frac{G(\xi)\,(1+\alpha h)^{-(1+\beta_1)}}{1+\beta_1} J_1(r\xi)\,\xi\,d\xi \tag{28}$$

We see that the equations (26) and (28) are identical thereby establishing the gerenal principle of reciprocity, i.e. the displacement at a depth h from a source is the same as the displacement at the surface from a source at depth h (Rayleigh J.W.S. 1945, Vol. I p, 150).

Surface displacements from a surface source

In order to study the surface displacement from a surface source, we put $h = 0$ in (28) and get

$$v = \frac{e^{ipt}}{\alpha c_s}\int_0^{\infty} \frac{G(\xi)}{1+\beta_1} J_1(r\xi)\,\xi\,d\xi$$

$$= \frac{e^{ipt}}{\alpha c_s}\int_0^{\infty} \frac{G(\xi)\,J_1(r\xi)\,\xi\,d\xi}{1+\sqrt{\{1+(1/\alpha^2 r_0^2)(\xi^2-p^2/v_{v0}^2)\}}} \tag{29}$$

Case I : Let $p > {}_1 v_{v0}$ where $\alpha_1 = \alpha r_0$

Then we have

$$v = \frac{r_0 e^{ipt}}{c_s}\int_0^{\infty} \frac{G(\xi)\,J_1(r\xi)\,\xi\,d\xi}{\alpha r_0 + \sqrt{\{\xi^2-(p^2/v_{v0}^2-\alpha^2 r_0^2)\}}}$$

$$= \frac{r_0 e^{ipt}}{c_s}\int_0^{\infty} \frac{G(\xi)\,J_1(r\xi)\,\xi\,d\xi}{\alpha_1+\sqrt{(\xi^2-k_1^2)}}$$

where

$$k_1 = \sqrt{\left(\frac{p^2}{v_{v0}^2} - \alpha_1^2\right)} \tag{30}$$

Let $g(r)$ be defined by

$$g(r) = \begin{cases} \dfrac{L}{r\sqrt{(b^2-r^2)}} & \text{for } 0 < r < b \\ 0 & \text{for } \quad r > b \end{cases} \tag{31}$$

so that (4) defines a finite circular torsional type source of radius b having discontinuities at $r = 0$ and $r = b$.

With the help of equations (4) and (31) we get

$$F = L\int_0^{b}\int_0^{\infty}\int_0^{2\pi} \frac{\delta(z-h)}{r\sqrt{(b^2-r^2)}}\, r\,dr\,d\theta\,dz$$

$$= \pi^2 L$$

Therefore the total force acting on the [illegible] region, [illegible]

$$G(\xi) = \int_0^{\infty} g(r)\, J_1(r\,\xi)\, r\, d\,r = L \int_0^{b} \frac{J_1(r\,\xi)\, dr}{\sqrt{(b^2 - r^2)}}$$

$$= \frac{L}{b\,\xi}\left[1 - \cos(b\,\xi)\right]$$

(Bateman p. 16)

and the displacement, v, is given by

$$v = \frac{L e^{ipt}}{b\, c_s} r_0 \int_0^{\infty} \frac{\{1 - \cos(b\,\xi)\}}{\alpha_1 + \sqrt{(\xi^2 - k_1^2)}}\, \xi\, J_1(r\,\xi)\, d\,\xi \tag{32}$$

Now using that result

$$J_1(r\,\xi) = -\frac{1}{\pi}\int_0^{\infty} \cosh u \left[e^{\,i\,r\,\xi\,\cosh u} + e^{-i\,r\,\xi\,\cosh u} \right] du \tag{33}$$

(Lamb, 1904, p. 34)

in (32) and changing the order of integration we get

$$v = -\frac{L\, e^{ipt}\, r_0}{b\, c_s\, \pi} \int_0^{\infty} \cosh(u)\, I(u)\, du, \tag{34}$$

where

$$I(u) = \int_{-\infty}^{\infty} \frac{1 - \cos(b\,\xi)}{\alpha_1 + \sqrt{(\xi^2 - k_1^2)}}\; e^{\,i\,r\,\xi\,\cosh u}\; d\,\xi \tag{35}$$

Evaluating I(u) and substituting in (34) we get

$$v = -\frac{\sqrt{2}\, k_1\, L\, r_0\, e^{i(pt+\pi/4)}}{b\, c_s\, \alpha_1^2\, \sqrt{\pi}} \left[I_1 - \frac{1}{2}\, e^{-ibk_1}\, I_2 - \frac{1}{2}\, e^{ibk_1}\, I_3 \right] \tag{36}$$

where

$$I_1 = \int_0^{\infty} \frac{e^{-ik_1\, r\, \cosh u}}{r\,(r\,\cosh u)^{3/2}}\, du$$

$$I_2 = \int_0^{\infty} \frac{\cosh(u)\, e^{-ik_1\, r\, \cosh u}}{(r\,\cosh u + b)^{3/2}}\, du \tag{37}$$

$$I_3 = \int_0^{\infty} \frac{\cosh(u)\, e^{-ik_1\, r\, \cosh u}}{(r\,\cosh u - b)^{3/2}}\, du$$

Applying the method of stationary phase, we have, for large r,

$$I_1 \approx \frac{1}{k_1 r^2}\sqrt{\frac{\pi}{2}}\,.\,e^{-i(k_1 r + \pi/4)}$$

$$I_2 \approx \sqrt{\left(\frac{\pi}{2k_1 r}\right)}\;\frac{e^{-i(k_1 r + \pi/4)}}{(r + b)^{3/2}} \qquad (38)$$

$$I_3 \approx \sqrt{\left(\frac{\pi}{2k_1 r}\right)}\;\frac{e^{-i(k_1 r + \pi/4)}}{(r-b)^{3/2}}$$

Using these values of I_1, I_2 and I_3, we get, for large values of r,

$$v = -\frac{L\, r_0}{b\, \bar{c}_s\, \alpha_1^2}\left[\frac{1}{r^2}\; e^{i(pt-k_1 r)} - \frac{1}{2\sqrt{r}\,(r+b)^{3/2}}\; e^{i(pt-(r+b)/k_1)} - \frac{1}{2\sqrt{r}\,(r-b)^{3/2}}\; e^{i(pt-(r-b)/k_1)}\right] \qquad (39)$$

Case II $p < \alpha_1 v_{v0}$

In this case we write

$$\sqrt{(\xi^2-k_1^2)} = \sqrt{(\xi^2+\bar{k}_1^2)} \qquad \text{where} \qquad \bar{k}_1 = \sqrt{\left(\alpha_1^2 - \frac{p^2}{v_{v0}^2}\right)}$$

Applying the method of contour integration, taking into account the contributions of the branch lines $\xi = \pm\, i\, k_1$ the displacement in the present case, is given by

$$v = -\frac{r_0\, \bar{k}_1\, L\, e^{ipt}}{b\, \bar{c}_s\, \alpha_1^2}\sqrt{\left(\frac{2}{\pi}\right)}\left[I_{11} - \tfrac{1}{2}\, e^{-b\bar{k}_1} I_{21} - \tfrac{1}{2}\, e^{b\bar{k}_1} I_{31}\right] \qquad (40)$$

where

$$I_{11} = \int_0^\infty \frac{e^{-r\bar{k}_1 \cosh u}}{\sqrt{(r^3 \cosh u)}}\, du$$

$$I_{21} = \int_0^\infty \frac{\cosh(u)\, e^{-r\bar{k}_1 \cosh u}}{(r \cosh u + b)^{3/2}}\, du \qquad (41)$$

$$I_{31} = \int_0^\infty \frac{\cosh(u)\, e^{-r\bar{k}_1 \cosh u}}{(r \cosh u - b)^{3/2}}\, du$$

Putting $\cosh u = 1 + \sigma$ and using Watson's lemma (Jeffreys, 1968, p. 501) for large $c_1 r$ we get

$$\text{[illegible]} \approx \frac{e^{-r\bar{c}_1}}{r^2}\sqrt{\frac{\pi}{2\bar{k}_1}}$$

$$I_{21} \approx \frac{e^{-r\bar{k}_1}}{(r+b)^{3/2}}\sqrt{\left(\frac{\pi}{2\bar{k}_1 r}\right)} \qquad (42)$$

$$I_{31} \approx \frac{e^{-r\bar{k}_1}}{(r-b)^{3/2}}\sqrt{\left(\frac{\pi}{2\bar{k}_1 r}\right)}$$

Substituting these values in (40) we get, for large values of r and $p < \alpha_1 v_{v0}$

$$v = -\frac{2 L r_0 e^{ipt}}{b \bar{c}_s \alpha_1^2}\left[\frac{e^{-r\bar{k}_1}}{r^2} - \frac{e^{-\bar{k}_1(r+b)}}{2\sqrt{r}(r+b)^{3/2}} - \frac{e^{-\bar{k}_1(r-b)}}{2\sqrt{r}(r-b)^{3/2}}\right] \quad (43)$$

Evaluation of the displacement v, for the homogeneous medium

It is seen that the displacement, v, for the homogeneous case, cannot be obtained directly by putting $\alpha = 0$ in (39) because the integrands in I_1, I_2, I_3 are no longer analytic near $\eta = 0$. For $\alpha = 0$, we, however, have

$$I_1 = i\int_0^\infty \frac{e^{-\eta r \cosh u}}{\sqrt{\eta}\,(i\eta - 2k_1)}\, d\eta$$

$$I_2 = i\int_0^\infty \frac{e^{-\eta(r \cosh u + b)}}{\sqrt{\eta\,(i\eta - 2k_1)}}\, d\eta \quad (44)$$

$$I_3' = i\int_0^\infty \frac{e^{-\eta(r \cosh u - b)}}{\sqrt{\eta\,(i\eta - 2k_1)}}\, d\eta$$

where, now, $k_1 = p/v_{v0}$. The function $1/\sqrt{(i\eta - 2k_1)}$ is analytic in the neighbourhood of the point $\eta = 0$ Evaluating (44) with the help of Watson's lemma and putting their values in (36) we get

$$v = \frac{i L r_0 e^{i(pt+\pi/4)}}{b \bar{c}_s}\sqrt{\left(\frac{2}{\pi k_1}\right)}\left[\frac{1}{\sqrt{r}}\int_0^\infty e^{-i r k_1 \cosh u}\sqrt{\cosh u}\, du\right.$$

$$-\frac{1}{2}\, e^{-i b k_1}\int_0^\infty \frac{e^{-i r k_1 \cosh u}}{\sqrt{(r \cosh u + b)}}\cosh u\, du \quad (45)$$

$$\left.-\frac{1}{2}\, e^{-i b k_1}\int_0^\infty \frac{e^{-i r k_1 \cosh u}}{\sqrt{(r \cosh u - b)}}\cosh u\, du\right]$$

Applying the method of stationary phase we get the displacement, v, for large values of r as

$$v = \frac{L r_0 v_{v0}}{b \bar{c}_s p}\left[\frac{1}{r}\, e^{-i(pt - pr/v_{v0} + \pi/2)}\right.$$

$$-\frac{1}{2\sqrt{(r^2 + r b)}}\cdot\; e^{i(pt - (r + b)p/v_{v0} + \pi/2)}$$

$$\left.-\frac{1}{2\sqrt{(r^2 - r b)}}\cdot\; e^{i(pt - (2 - b)p/v_{v0} + \pi/2)}\right] \quad (46)$$

For the homogeneous isotropic medium we have

$$\bar{c}_4 = \bar{c}_{44} = \mu_0,\ r_0 = 1,\ v_{v0} = \beta$$

$$v = \frac{L\beta}{b\,p\,\mu_0}\left[\frac{1}{r}\,e^{i\,(pt - pr/\beta + \pi/2)} - \frac{1}{2\sqrt{r\,(r+b)}}\,e^{i\,(pt - p\,(r+b)/\beta + \pi/2)} - \frac{r}{2\sqrt{r\,(r-b)}}\,e^{i\,(pt - p\,(r-b)/\beta + \pi/2)}\right]$$

which is the same as obtained by Sidhu (1971).

Physical interpretation

For $p > \alpha_1 v_{v0}$, the equation (39) shows that the displacement at a distant point in the medium consists of three phases. First term corresponds to a phase emitting from the centre of the source and the amplitude is inversely proportional to the square of its distance from the centre. The second term corresponds to a phase emitting from the farthest end of the source and the amplitude varies as $\frac{1}{r^{1/2}\,(r+b)^{3/2}}$. The third term indicates the contribution from the nearest end of the source and its amplitude varies as $\frac{1}{r^{1/2}\,(r-b)^{3/2}}$. Comparing these results with the corresponding homogeneous anisotropic and the homogeneous isotropic media, we find that the decay of amplitudes is rapid in the case of inhomogeneous anisotropic media.

For the case $p < \alpha_1 v_{v0}$ the displacement at a large distance r is the contribution of three phases emitting from the centre of the farthest end and the nearest end of the source and the decay of their corresponding amplitudes varies as

$$\frac{1}{r^2 \exp\,(\bar{c}_1\,r)},\quad \frac{1\,\exp\,(-k_1\,(r+b))}{r^{1/2}\,(r+b)^{3/2}}$$

and

$$\frac{1}{r^{1/2}\,(r-b)^{3/2}}\,\exp\,(-k_1\,(r-b))$$

which is more rapid than that for the case $p > \alpha_1 v_{v0}$.

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