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COUPLED VIBRATIONS OF A PRETWISTED SLENDER BEAM UNDER HARMONIC EXCITATION

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Introduction

The analysis presented in this paper considers coupled bending-bending torsional vibrations of a pretwisted slender cantilever beam, excited by the periodic motion of its supporting base as shown in the figure 1. The beam is pretwisted linearly about the centroidal axis having maximum angle of pretwist α at the free end. The shear centre of each cross section of the beam does not coincide with the centre of gravity, consequently torsional and bending-bending oscillations are coupled.

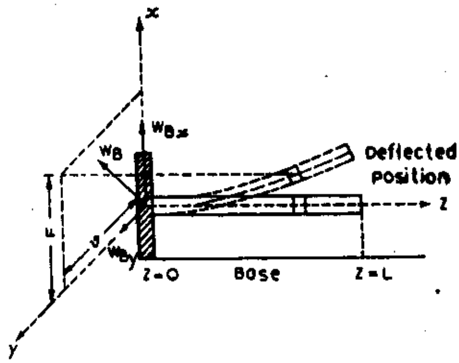

Fig. 1 Beam with harmonic excitation

The Differential Equations

The differential equation for the deflected form of the neutral axis of a bar according to the elementary theory of bending is

$$\frac{\partial^2}{\partial z^2}\left(EI \frac{\partial^2 w}{\partial z^2} \right) = \overline{w} \tag{1}$$

where $\overline{w}$ is the intensity of the distributed load along Z axis and w is the deflection prependicular to z-axis.

Since the blade is pretwisted, the deflections occur in x as well as in y-directions resulting in two such differential equations of the form

$$\frac{\partial^2}{\partial z^2}\left(EI'_{yy} \frac{\partial^2 u}{\partial z^2} + EI'_{xy} \frac{\partial^2 v}{\partial z^2} \right) = \overline{w_1} \tag{2a}$$

and

$$\frac{\partial^2}{\partial z^2}\left(EI'_{xx} \frac{\partial^2 v}{\partial z^2} + EI'_{xy} \frac{\partial^2 u}{\partial z^2} \right) = \overline{w_2} \tag{2b}$$

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If the load is distributed along the centrodial axis, the given load can be replaced by the same load distributed along the shear centre axis, and a torque of intensity $(\overline{w_1}\,\delta_y + \overline{w_2}\,\delta_x)$ distributed along the same axis.

Since the torsion is not uniform the relation between the variable torque T and the angle of twist θ is given by Timoshenko (1955)

$$T = GJ\frac{\partial\theta}{\partial z} - C_1\frac{\partial^3\theta}{\partial z^3}$$

where GJ is the torsional rigidity and c_1 is the warping rigidity.

Differentiation of this equation with respect to z gives

$$GJ\frac{\partial^2\theta}{\partial z^2} - C_1\frac{\partial^4\theta}{\partial z^4} = (\overline{w_1}\,\delta_y + \overline{w_2}\,\delta_x) \tag{3}$$

For a vibrating bar the intensity of inertia force is

$$-m\frac{\partial^2}{\partial t^2}(u + \delta_y\theta) \quad \text{and} \quad -m\frac{\partial^2}{\partial t^2}(v + \delta_x\theta)$$

respectively in x and y directions and the intensity of the inertia moment about z-axis is

$$-I_\theta\frac{\partial^2\theta}{\partial t^2}$$

The following differential equations for the coupled bending-bending-torsion vibrations are obtained by replacing the statical loads in (2a, 2b) and (3) by the inertia forces

$$\frac{\partial^2}{\partial z^2}\left(EI'_{yy}\frac{\partial^2 u}{\partial z^2} + EI'_{xy}\frac{\partial^2 v}{\partial z^2}\right) = -m\frac{\partial^2}{\partial t^2}(u + \delta_y\theta) \tag{4a}$$

$$\frac{\partial^2}{\partial z^2}\left(EI'_{xx}\frac{\partial^2 v}{\partial z^2} + EI'_{xy}\frac{\partial^2 u}{\partial z^2}\right) = -m\frac{\partial^2}{\partial t^2}(v + \delta_x\theta) \tag{4b}$$

$$GJ\frac{\partial^2\theta}{\partial z^2} - c_1\frac{\partial^4\theta}{\partial z^4} = m\,\delta_y\frac{\partial^2}{\partial t^2}(u+\delta_y{}^*\theta) + m\,\delta_x\frac{\partial^2}{\partial t^2}(v+\delta_x{}^*\theta) + I_\theta\frac{\partial^2\theta}{\partial t^2} \tag{4c}$$

where

$$I'_{yy} = \frac{I_{xx}+I_{yy}}{2} - \frac{I_{xx}-I_{yy}}{2}\cos 2\alpha\frac{z}{L} - I_{xy}\,\mathrm{Sin}\,2\alpha\frac{z}{L}$$

$$I'_{xx} = \frac{I_{xx}+I_{yy}}{2} + \frac{I_{xx}-I_{yy}}{2}\cos 2\alpha\frac{z}{L} + I_{xy}\,\mathrm{Sin}\,2\alpha\frac{z}{L}$$

$$I'_{xy} = \frac{I_{xx}-I_{yy}}{2}\,\mathrm{Sin}\,2\alpha\frac{z}{L} + I_{xy}\cos 2\alpha\frac{z}{L}$$

$$\delta_x = \delta_{xo}\cos\alpha\frac{z}{L} - \delta_{yo}\sin\alpha\frac{z}{L}$$

$$\delta_y = \delta_{yo}\cos\alpha\frac{z}{L} - \delta_{xo}\sin\alpha\frac{z}{L}$$

are taken from Carnegie (1957).

These equations are the same as obtained by Carnegie (1962) using energy method except the * terms.

The right hand side of equations (4a) and (4b) represents an inertia loading, and these equations must be modified by considering the base motion w_{B_x} and w_{B_y} in x and y directions respectively as Tseng and Dugundji (1970) the governing differential equations then becomes

$$\frac{\partial^2}{\partial z^2}\left(EI'_{yy}\frac{\partial^2 u}{\partial z^2}+EI'_{xy}\frac{\partial^2 v}{\partial z^2}\right)=-m\frac{\partial^2}{\partial t^2}(u+\delta_y\theta)-m\frac{\partial^2 w_{B_x}}{\partial t^2} \tag{6a}$$

$$\frac{\partial^2}{\partial z^2}\left(EI'_{xx}\frac{\partial^2 v}{\partial z^2}+EI'_{xy}\frac{\partial^2 u}{\partial z^2}\right)=-m\frac{\partial^2}{\partial t^2}(v+\delta_x\theta)-m\frac{\partial^2 w_{B_y}}{\partial t^2} \tag{6b}$$

$$GJ\frac{\partial^2\theta}{\partial z^2}-c_1\frac{\partial^4\theta}{\partial z^4}=m\delta_y\frac{\partial^2}{\partial t^2}(u+\delta_y\theta)+m\delta_x\frac{\partial^2}{\partial t^2}(v+\delta_x\theta)+I_\theta\frac{\partial^2\theta}{\partial t^2} \tag{6c}$$

Determination of Natural Frequencies

The solutions of equations (6) are of the form

$$\begin{aligned} u(z,t) &= A\,U(z)\,e^{i\omega t}\\ v(z,t) &= B\,V(z)\,e^{i\omega t}\\ w_{B_x} &= A\,F_o\,e^{i\omega t}\\ w_{B_y} &= B\,F_o\,e^{i\omega t}\\ \theta(z,t) &= C\,\beta(z)\,e^{i\omega t}\end{aligned} \tag{7}$$

Where F_o is the forcing amplitude, A, B and C are constants which are not independent and U(z), V(z) and $\beta(z)$ are functions of z only.

The function U(z), V(z), $\beta(z)$ satisfy all the boundary conditions of the beam which are as follows

$$\begin{aligned} &u=v=\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=\theta=\frac{\partial^2\theta}{\partial z^2}=0 \qquad \text{at } Z=0\\ &\frac{\partial^2 u}{\partial z^2}=\frac{\partial^2 v}{\partial z^2}=\frac{\partial^3 u}{\partial z^3}=\frac{\partial^3 v}{\partial z^3}=\frac{\partial\theta}{\partial z}=\frac{\partial^3\theta}{\partial z^3}=0 \quad \text{at } Z=L\end{aligned} \tag{8}$$

Substitution of equations (7) in (6) gives

$$\left[\frac{d^2}{dz^2}\left(EI'_{yy}\frac{d^2U}{dz^2}\right)-m(U+F_o)\,\omega^2\right]A+\frac{d^2}{dz^2}\left(EI'_{xy}\frac{d^2V}{dz^2}\right)B-m\delta_y\,\omega^2\,\beta\,C=0 \tag{9a}$$

$$\frac{d^2}{dz^2}\left(EI'_{xy}\frac{d\,U}{dz^2}\right)A+\left[\frac{d^2}{dz^2}\left(EI'_{xx}\frac{d^2V}{dz^2}\right)-m(V+F_o)\,\omega^2\right]B-m\delta_x\,\omega^2\,\beta\,C=0 \tag{9b}$$

$$-m\delta_y\,\omega^2\,UA-m\delta_x\,\omega^2\,VB+\left[-\left\{GJ\frac{d^2\beta}{dz^2}-c_1\frac{d^4\beta}{dz^4}\right\}-\left\{I_\theta+m(\delta_x^2+\delta_y^2)\right\}\omega^2\beta\right]C= \tag{9c}$$

The equations are now put in terms of dimensionless variables $\xi=z/L$, $f=U/L$, $\phi=V/L$, $\psi=\beta/L$, and substituting values of I'_{yy}, I'_{xx}, I'_{xy}.

Equations (9) becomes

$$\left[\frac{E}{mL^4}\left\{\frac{I_{xx}+I_{yy}}{2}\frac{d^4f}{d\xi^4}-\frac{I_{xx}-I_{yy}}{2}\frac{d^2}{d\xi^2}\left(\cos 2\alpha\xi\,\frac{d^2f}{d\xi^2}\right)-I_{xy}\frac{d^2}{d\xi^2}\left(\sin\,2\alpha\xi\,\frac{d^2f}{d\xi^2}\right)\right\}\right.$$

$$-\omega^2\left(f+\frac{F_e}{L}\right)A-\frac{E}{mL^4}\left[\frac{I_{xx}-I_{yy}}{2}\frac{d^2}{d\xi^2}\left(\sin 2\alpha\xi\frac{d^2\phi}{d\xi^2}\right)-I_{xy}\frac{d^2}{d\xi^2}\left(\cos 2\alpha\xi\frac{d^2\phi}{d\xi^2}\right)\right]B$$
$$-\omega^2\delta_y\psi C=0 \qquad (10a)$$

$$-\frac{E}{mL^4}\left[\frac{I_{xx}-I_{yy}}{2}\frac{d^2}{d\xi^2}\left(\sin 2\alpha\xi\frac{d^2f}{d\xi^2}\right)-I_{xy}\left(\cos 2\alpha\xi\frac{d^2f}{d\xi^2}\right)\right]A+\left[\frac{E}{mL^4}\left\{\frac{I_{xx}+I_{yy}}{2}\frac{d^4\phi}{d\xi^4}\right.\right.$$
$$\left.\left.+\frac{I_{xx}-I_{yy}}{2}\frac{d^2}{d\xi^2}\left(\cos 2\alpha\xi\frac{d^2\phi}{d\xi^2}\right)+I_{xy}\frac{d^2}{d\xi^2}\left(\sin 2\alpha\xi\frac{d^2\phi}{d\xi^2}\right)\right\}-\omega^2\left(\phi+\frac{F_e}{L}\right)\right]B$$
$$-\omega^2\delta_x\psi C=0 \qquad (10b)$$

$$-\delta_y\omega^2 f A-\delta_x\omega^2\phi B+\left[-\left\{\frac{GJ}{mL^2}\frac{d^2\psi}{d\xi^2}-\frac{c_1}{mL^4}\frac{d^4\psi}{d\xi^4}\right\}-\left(\frac{I_e}{m}+\delta_x^2+\delta_y^2\right)\omega^2\psi\right]C=0 \qquad (10c)$$

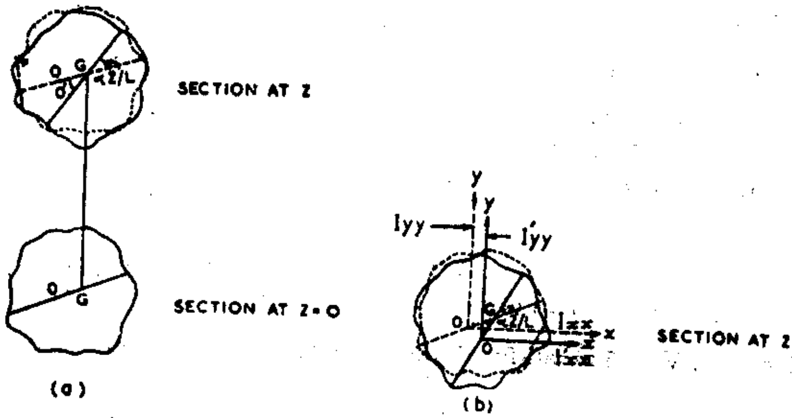

Fig. 2 Geometry of the system

For an approximate determination of the fundamental frequency $f(\xi)$, and $\phi(\xi)$ are chosen as the shape function for the fundamental mode of uncoupled bending vibration and $\psi(\xi)$ as the shape function for the fundamental mode of uncoupled torsional vibration of a uniform cantilever beam. These shape functions satisfy the boundary conditions (8) and are

$$f(\xi)=\phi(\xi)=\cosh\lambda\xi-\cos\lambda\xi-\sigma r\,(\sinh\lambda\xi-\sinh\lambda\xi,\quad \psi(\xi)=\sin(\pi/2)\,\xi \qquad (11)$$

where $\lambda = 1{\cdot}87510,\quad \sigma r = 0{\cdot}7341$

Equations (10) can be solved for ω^2 but the result is a function of ξ, since f, ϕ and ψ are not the exact shape functions. This difficulty can be overcome Fung (1955) by multiplying (10 a) by f, (10b) by ϕ and (10c) by ψ and integrating with respect to ξ from 0 to 1. The method results in the familiar Rayleigh quotient Collatz (1960) when applied to uncoupled problems, and is an extension of Rayleigh's method to the coupled problem. The following equations are obtained

$$\begin{aligned}
(a_1-a_2\omega^2)A-a_3B-a_4\omega^2C&=0\\
-a_5A+(a_6-a_7\omega^2)B-a_8\omega^2C&=0\\
-a_9\omega^2A-a_{10}\omega^2B+(a_{11}-a_{12}\omega^2)C&=0
\end{aligned} \qquad (12)$$

where

$$a_1 = \frac{E}{mL^4}\left[\frac{I_{xx}+I_{yy}}{2}\int_0^1 \frac{d^4f}{d\xi^4} f d\xi - \frac{I_{xx}-I_{yy}}{2}\int_0^1 \frac{d^2}{d\xi^2}\left(\cos 2\alpha\xi \frac{d^2f}{d\xi^2}\right) f d\xi - I_{xy}\int_0^1 \frac{d^2}{d\xi^2}\left(\sin 2\alpha\xi \frac{d^2f}{d\xi^2}\right) f d\xi\right]$$

$$a_2 = \int_0^1 \left(f^2 + \frac{F_o}{L} f\right) d\xi$$

$$a_3 = \frac{E}{mL^4}\left[\frac{I_{xx}-I_{yy}}{2}\int_0^1 \frac{d^2}{d\xi^2}\left(\sin 2\alpha\xi \frac{d^2\phi}{d\xi^2}\right) f d\xi - I_{xy}\int_0^1 \frac{d^2}{d\xi^2}\left(\cos 2\alpha\xi \frac{d^2\phi}{d\xi^2}\right) f d\xi\right]$$

$$a_4 = a_9 = \int_0^1 \delta_y f \psi d\xi$$

$$a_5 = \frac{E}{mL^4}\left[\frac{I_{xx}-I_{yy}}{2}\int_0^1 \frac{d^2}{d\xi^2}\left(\sin 2\alpha\xi \frac{d^2f}{d\xi^2}\right) \phi d\xi - I_{xy}\int_0^1 \frac{d^2}{d\xi^2}\left(\cos 2\alpha\xi \frac{d^2f}{d\xi^2}\right) \phi d\xi\right]$$

$$a_6 = \frac{E}{mL^4}\left[\frac{I_{xx}+I_{yy}}{2}\int_0^1 \frac{d^4\phi}{d\xi^4} \phi d\xi + \frac{I_{xx}-I_{yy}}{2}\int_0^1 \frac{d^2}{d\xi^2}\left(\cos 2\alpha\xi \frac{d^2\phi}{d\xi^2}\right) \phi d\xi - I_{xy}\int_0^1 \frac{d^2}{d\xi^2}\left(\sin 2\alpha\xi \frac{d^2\phi}{d\xi^2}\right) \phi d\xi\right]$$

$$a_7 = \int_0^1 \left(\phi^2 + \frac{F_o}{L} \phi\right) d\xi$$

$$a_8 = a_{10} = \int_0^1 \delta_x \phi\psi d\xi$$

$$a_{11} = -\left\{\frac{GJ}{mL^2}\int_0^1 \frac{d^2\psi}{d\xi^2} \psi d\xi - \frac{C_1}{mL^4}\int_0^1 \frac{d^4\psi}{d\xi^4} \psi d\xi\right\}$$

$$a_{12} = \int_0^1 \left(\frac{I_z}{m} + \delta_x^2 + \delta_y^2\right)\psi^2 d\xi$$

For a non trivial solution A, B and C must not vanish consequently the determinant of the coefficients of equations (12) must be zero.

$$\begin{vmatrix} a_1+a_2\,\omega^2 & -a_3 & -a_4\,\omega^2 \\ -a_5 & a_6-a_7\,\omega^2 & -a_8\,\omega^2 \\ -a_9\,\omega^2 & -a_{10}\,\omega^2 & a_{11}-a_{12}\,\omega^2 \end{vmatrix} = 0 \qquad (13)$$

This gives a third degree polynomial in ω^2. The smallest of the three values of ω^2 given by (13) is an upper bound for the frequency of the fundamental mode of coupled vibrations. The other higher values of ω^2 are the upper bounds for the next two higher modes of vibrations. To obtain the smallest value of ω^2, we use Newton's iterative method till it shows convergence.

Numerical Example

A numerial example for the coupled bending-bending-torsional vibrations of a pretwisted beam under harmonic excitation is now presented. The frequencies are computed from (13), and the cross section of the beam is taken as semicircle of radius a_o and thickness t_1. The physical constants are as follows :

$L = 10$ in, $E = 30\times10^6$ lb/in^2, $G = 12 \times 10^6$ lb/in^2.

$a_3 = \cdot45$ in, $t_1 = .25$ in, $m = \cdot09818$ lb.

$$I_{xx} = \tfrac{1}{2}\,\pi\, a_o^3\, t_1, \quad I_{yy} = \pi\, a_o\, t_1 \left\{ \frac{a_o^2}{2} + \left(\frac{2a_o}{\pi}\right)^2 + \delta_{x_o}^{\,2} \right\}$$

$$I_{xy} = 0, \quad c_1 = E a_o^5\, t_1 \left(\frac{3}{12} - \frac{8}{\pi} \right), \quad J = \frac{\pi\, a_o\, t_1^3}{3}$$

$$I_\theta = m \left\{ \frac{a_o^2}{2} + \left(\frac{2a_o}{\pi}\right)^2 + \delta_{x_o}^{\,2} \right\}, \quad \delta_{x_o} = \frac{2a_o}{\pi}, \; \delta_{y_o} = 0.$$

With these values, the fundamental frequencies for various values of α and F_o/L have been tabulated below. In Fig. (3) $\frac{\omega_1}{\omega_{\alpha=0}}$ has been plotted as a function of α and in Fig. (4) $\frac{\omega_1}{\omega_{F_o/L=0}}$ has been plotted as a function of F_o/L.

Discussion and Conclusion

The equations of motion for bending-bending and torsional vibrations of a pretwisted beam which have been obtained using a beam type theory are similar to those obtained earlier by Carnegie (1962) using energy principles. However, the equations in the paper include higher order terms also and therefore expected to be more accurate. The method based on Rayleigh's Quotient which has been used to solve the equations of motion of the complicated system under consideration to give an upper bound of the fundamental frequency of the coupled vibrations is also expected to give reasonably accurate results as the accuracy and the convergence of the method has already been investigated by the authors Tomar and Dhole (1975) specially when δ_x and δ_y are small.

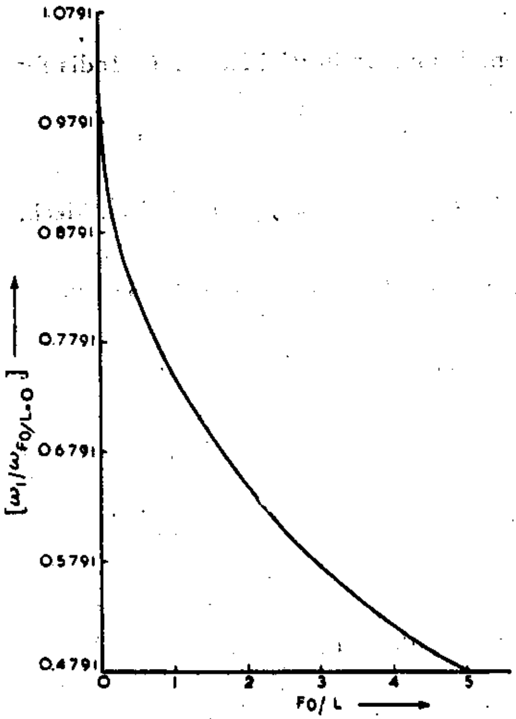

Fig. 3 Effect of pretwist on frequency

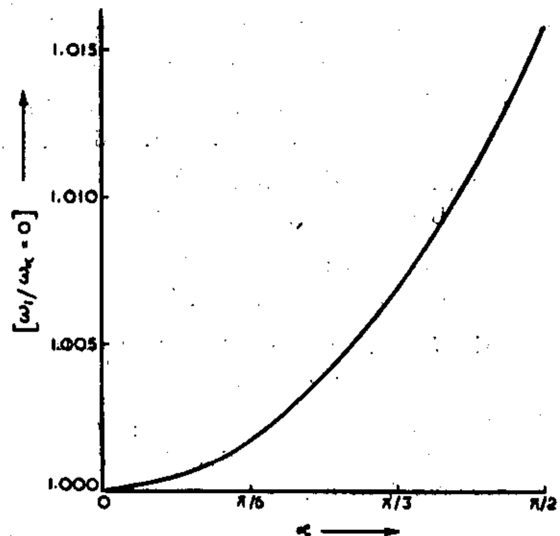

Fig. 4 Effect of forcing amplitude on frequency

The results obtained for the fundamental frequency of coupled torsional and bending-bending vibrations show that it increases as the angle of pretwist α increases and decreases as the ratio of forcing amplitude to blade length increases.

TABLE

Square of the fundamental frequency i.e. ω_1^2

α in radians	For fixed Fo/L=2	$\frac{Fo}{L}$	For fixed $\alpha=\pi/6$ radians
0	4.33598×10^3	0	10.20583×10^3
		1	6.12083×10^3
$\frac{\pi}{6}$	4.36491×10^3	2	4.36491×10^3
		3	3.38932×10^3
$\frac{\pi}{3}$	4.41743×10^3	4	2.76966×10^3
$\frac{\pi}{2}$	4.58253×10^3	5	2.34133×10^3

Acknowledgements

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APPENDIX

Notation

GG = Longitudnal axis through the centroids of the cross sections of the beam about which it is uniformly pretwisted from angle zero at the fixed end to angle α at the free end. The distance z is measured along this axis.

ox, oy, oz = Co-ordinate axis through the shear centre of the cross section at fixed end i.e. at $z=0$, oz is parallel to GG axis.

o′x, o′y, o′z = Co-ordinate axis through the shear centre o′ of the cross section at the point z and parallel to ox, oy and oz axis respectively.

W_B = Base motion perpendicular to the axis of the beam and in the xy-plane.

u = deflection of the shear centre of the cross section at the point z of the beam in x-direction.

v = deflection of the shear centre of the cross section at the point z of the beam in y-direction.

L = length of the beam.

E = modulus of elasticity.

G = shear modulus of rigidity.

C_1 = warping rigidity.

I_{xx} = second moment of area at $z = 0$ about x-axis.

I_{yy} = second moment of area at $z = 0$ about y-axis.

I_{xy} = product moment of area of cross section at $z = 0$ about ox and oy-axis.

I'_{xx} = second moment of area of cross section at the point z of the beam about o'x axis.

I'_{yy} = second moment of area of cross section at the point z of the beam about o'y axis.

I'_{xy} = product moment of area of cross section at the point z of the beam about o'x and o'y axis.

I_θ = mass moment of inertia about the shear centre per unit length of the beam.

δ_{xo} = Co-ordinate distance between shear centre and centroid in x-direction of the cross section at $z = 0$ of the beam.

δ_{yo} = Co-ordinate distance between shear centre and centroid in y-direction of the cross section at $z = 0$ of the beam.

δ_x = Co-ordinate distance between shear centre and centroid in x-direction of the cross section at the point z of the beam.

δ_y = Co-ordinate distance between shear centre and centroid in y-direction of the cross section at the point z of the beam.

J = a constant depending upon the cross section of the beam such that GJ is the torsional rigidity.

α = the angle of pretwist in radians of the section at $Z = L$.

t = time

ξ = dimensionless variable $\xi = z/L$.

ω = frequency of vibration.