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THE EFFECT OF ROTATORY INERTIA AND SHEAR DEFORMATION ON THE VIBRATIONS OF BEAM ON ELASTIC FOUNDATION

Y. C. Das* and G. N. Harper†

Summary

A theoretical analysis of the effects of transverse shear and rotatory inertia on the natural frequencies of a uniform beam on an elastic foundation is presented. Equations for frequencies and modes are derived for six combinations of simple end conditions. For the case of a fixed-free beam on an elastic foundation numerical results are presented in the from of curves giving frequencies in terms of the parameters for shear deflection, rotatory inertia, and foundation modulus.

Introduction

It is well known that the classical, one-dimensional Bernoulli-Euler theory of flexural motions of elastic beams is inadequate for the study of higher modes of beams, as well as for the modes of beams for which the cross-sectional dimensions are not small when compared to their length between modal sections. Rayleigh (1) introduced the effect of rotatory inertia and Timoshenko (2,3) extended the theory to include the effect of transverse shear deformation. Others who have contributed to an understanding of the importance of these effects are : Jacobsen (4), Searle (5), Kruszewski (6), Sutherland and Goodman (7), Anderson (8), Dolph (9), Minclin and Deresiewicz (10), Herrmann (11), and Huang (12, 13).

The square of the frequncies of a beam on an elastic, Winkler-type fundation can be obtained from the square of the frequencies of beam *in vocuo* by adding a suitable foundation parameter (14). This is not possible if the effects of shear deformation and rotatory inertia are introduced into the classical theory of vibrations of beams on elastic foundation. A. I. Tseitlin (15) has considered these effects on the vibrations of beams of infinite length on an elastic foundation. The present paper deals with frequency equations and normal modes of flexural vibrations of beams of finite length on an elastic, Winkler-type foundation, including the effects of shear deflection and rotatory inertia for various cases of simple end conditions. Solutions are obtained for total deflection and bending slope. The frequency and normal mode equations are derived for six common types of end conditions. A numerical example is given for fixed-free end conditions.

Statement of the Problem

The differential equations of motion for a beam on a Winkler-type foundation with shear deformation and rotatory inertia taken into account are of the from

$$EI\frac{\partial^2\psi}{\partial x^2} + kAG\left(\frac{\partial y}{\partial x} - \psi\right) - \frac{I\gamma}{g}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{1}$$

$$\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - kAG\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial\psi}{\partial x}\right) + Ky = 0 \tag{2}$$

where

E = modulus of elasticity

G = shear modulus of elasticity

* Associate Professor, Dept. of Civil Engineering, Indian Institute of Technology, Kanpur, India
† Instructer Dept. of Civil Engineering, University of Illinois, Urbana, Illinois, U.S.A.

I = area moment of inertia of section
A = cross-sectional area
γ = weight per unit volume
k = numerical shape factor for the cross section
K = coefficient of the foundation
y = total deflection
ψ = bending slope

Eliminating y or ψ from equations (1) and (2), one obtains the following two uncoupled equations in y and ψ

$$EI\frac{\partial^4\psi}{\partial x^4}-\frac{EIK}{kAG}\frac{\partial^2\psi}{\partial x^2}-\left(\frac{\gamma I}{g}+\frac{EI}{gk}\frac{\gamma}{G}\right)\frac{\partial^4\psi}{\partial x^2\partial t^2}+\left(\frac{\gamma A}{g}+\frac{I}{gk}\gamma\frac{K}{AG}\right)\frac{\partial^2\psi}{\partial t^2}+\frac{\gamma I}{g}\frac{\gamma}{gkG}\frac{\partial^4\psi}{\partial t^4}+K\psi=0 \tag{3}$$

$$EI\frac{\partial^4 y}{\partial x^4}-\frac{EIK}{kAG}\frac{\partial^2 y}{\partial x^2}-\left(\frac{\gamma I}{g}+\frac{EI}{gk}\frac{\gamma}{G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2}+\left(\frac{\gamma A}{g}+\frac{I\gamma}{gk}\frac{K}{AG}\right)\frac{\partial^2 y}{\partial t^2}+\frac{\gamma I}{g}\frac{\gamma}{gkG}\frac{\partial^4 y}{\partial t^4}+Ky=0 \tag{4}$$

The shear slope, moment and shear are given by:

$$\text{Shear Slope};\ \phi(x,t)=\frac{\partial y}{\partial x}-\psi \tag{5}$$

$$\text{Moment};\ M(x,t)=-EI\frac{\partial\psi}{\partial x} \tag{6}$$

$$\text{Shear};\ Q(x,t)=kAG\left(\frac{\partial y}{\partial x}-\psi\right) \tag{7}$$

For the simplest end configurations, the boundary conditions are the following:

$$\text{Hinged End};\ y=0 \text{ and } \frac{\partial\psi}{\partial x}=0 \tag{8}$$

$$\text{Clamped End};\ y=0 \text{ and } \psi=0 \tag{9}$$

$$\text{Free End};\ \frac{\partial\psi}{\partial x}=0 \text{ and } \frac{\partial y}{\partial x}-\psi=0 \tag{10}$$

Frequency Equations and Modal Forms

Let us take the solutions of equations (1) to (4) in the form

$$y(x,t)=Y(\xi)\,e^{ipt} \tag{11}$$

$$\psi(x,t)=\varphi(\xi)\,e^{ipt} \tag{12}$$

with

$$\xi=x/L \tag{13}$$

where

$i=\sqrt{-1}$
p = angular frequency (restricted to real number)
L = length of the beam

Omitting the factor e^{ipt},

$$s^2 \varphi'' - (1 - b^2r^2s^2)\varphi + \frac{Y'}{L} = 0 \tag{14}$$

$$Y'' + s^2(b^2 - q)Y - L\varphi' = 0 \tag{15}$$

$$\varphi^{iv} + [b^2(r^2 + s^2) - s^2q]\varphi'' - [b^2(1 - b^2r^2s^2) - q(1 - b^2r^2s^2)]\varphi = 0 \tag{16}$$

$$Y^{iv} + [b^2(r^2 + s^2) - s^2q]Y'' - [b^2(1 - b^2r^2s^2) - q(1 - b^2r^2s^2)]Y = 0 \tag{17}$$

where

$$b^2 = \frac{1}{EI}\frac{\gamma A}{g}L^4p^2 \tag{18}$$

$$r^2 = \frac{I}{AL^2} \tag{19}$$

$$s^2 = \frac{EI}{kAG\,L^2} \tag{20}$$

$$q = \frac{KL^4}{EI} \tag{21}$$

The dimensionless parameter b is directly related to the frequencies of vibration, p The dimensionless parameters r, s, and q are measures of the effects of rotatory inertia, shear deformation, and the elastic foundation, respectively. Solutions of equations (16) and (17) may be found to be

$$Y = C_1 \cosh \alpha\xi + C_2 \sinh \alpha\xi + C_3 \cos \beta\xi + C_4 \sin \beta\xi \tag{22}$$

$$\varphi = C'_1 \cosh \alpha\xi + C'_2 \sinh \alpha\xi + C'_3 \cos \beta\xi + C'_4 \sin \beta\xi \tag{23}$$

where

$$\alpha = \frac{1}{\sqrt{2}}\left[\left\{[b^2(r^2 - s^2) + s^2q]^2 + 4(b^2 - q)\right\}^{1/2} - \left\{b^2(r^2 + s^2) - s^2q\right\}\right]^{1/2} \tag{24}$$

$$\beta = \frac{1}{\sqrt{2}}\left[\left\{[b^2(r^2 - s^2) + s^2q]^2 + 4(b^2 - q)\right\}^{1/2} + \left\{b^2(r^2 + s^2) - s^2q\right\}\right]^{1/2} \tag{25}$$

The eight constants in equations (22) and (23) are not all independent, but are related by the equations (14) or (15) as follows :

$$C'_1 = \left\{\frac{\alpha^2 + s^2(b^2 - q)}{L\alpha}\right\}C_2 \tag{27}$$

$$C'_2 = \left\{\frac{\alpha^2 + s^2(b^2 - q)}{L\alpha}\right\}C_1 \tag{28}$$

$$C'_3 = -\left\{\frac{s^2(b^2 - q) - \beta^2}{L\beta}\right\}C_4 \tag{29}$$

$$C'_4 = \left\{\frac{s^2(b^2 - q) - \beta^2}{L\beta}\right\}C_3 \tag{30}$$

It should be noted that the solutions in equations (22) and (23) apply only under the the conditions

$$[b^2(r^2 - s^2) + s^2q]^2 + 4(b^2 - q) > 0 \tag{26a}$$

and

$$\{[b^2(r^2 - s^2) + s^2q]^2 + 4(b^2 - q)\}^{1/2} \pm [b^2(r^2 + s^2) - s^2q] > 0 \tag{26b}$$

Inequality (26a) is essentially a requirement that α and β shall not be complex. If inequality (26a) is violated, frequencies for the boundary conditions given later become complex. Since only real frequencies are physically of interest, b will always be chosen such that both the inequality (26a) and the frequency equation for the particular boundary condition are satisfied.

With b chosen so that (26a) is satisfied, (26b) requires that α and β be real. Violation of (26b) means that either α or β becomes a pure imaginary. (The forced satisfaction of (26a) precludes α and β becoming imaginary simultaneously). It is still possible, however, to have real frequencies, even with α or β imaginary. Suppose that either α or β becomes imaginary; equations (24) and (25) then become

$$\alpha = \frac{i}{\sqrt{2}}\left[-\left\{[b^2(r^2-s^2)+s^2q]^2+4(b^2-q)\right\}^{1/2}+\left\{b^2(r^2+s^2)-s^2q\right\}\right]^{1/2} = i\,\alpha' \tag{24a}$$

or

$$\beta = \frac{i}{\sqrt{2}}\left[-\left\{[b^2(r^2-s^2)+s^2q]^2+4(b^2-q)\right\}^{1/2}+\left\{b^2(r^2+s^2)-s^2q\right\}\right]^{1/2} = i\,\beta' \tag{25a}$$

Substitution of either α or β according to (24a) and (25a) in equations (22) and (23) then gives the solutions in terms of the real α' and β':

For α imaginary

$$Y = C_1\cos\alpha'\xi + C_2\sin\alpha'\xi + C_3\cos\beta\xi + C_4\sin\beta\xi \tag{22a}$$

$$\varphi = C'_1\cos\alpha'\xi + C'_2\sin\alpha'\xi + C'_3\cos\beta\xi + C'_4\sin\beta\xi \tag{23a}$$

and for β imaginary

$$Y = C_1\cosh\alpha\xi + C_2\sinh\alpha\xi + C_3\cosh\beta'\xi + C_4\sinh\beta'\xi \tag{22b}$$

$$\varphi = C'_1\cosh\alpha\xi + C'_2\sinh\alpha\xi + C'_3\cosh\beta'\xi + C'_4\sinh\beta'\xi \tag{23b}$$

where again the eight constants of (22a) and (23a), as well as (22b) and (23b), are related through equations similar to equations (27 — 30).

For the homogeneous boundary conditions described by equations (8), (9) and (10), only the ratios of constants C_1, C_2, C_3 and C_4 or C'_1, C'_2, C'_3, and C'_4 are determinate. Application of any combination of the boundary conditions leads to the characteristic frequency equation. The roots of this equation are the frequency values, b, for which a nontrivial solution is valid. For each value of b, there is a corresponding natural mode.

The frequency equations and natural modes are given below for six common combinations of the boundary conditions (8), (9) and (10). In all six cases it has been assumed that the solutions to the differential equations (16) and (17) are those given in equations (22) and (23). As mentioned above, however, violation of inequality (26b) will change the form of the solution to that given in (22a) and (23a); the α and β of equations (24a) and (25a) should then be substituted into the frequency and mode equations given below to obtain the real frequencies and modes. An example of this substitution for the fixed-free beam, Case 5, is contained in the numerical problem. Frequency equations and natural modes are now given for the followinng six support conditions.

Case (1): Clamped at $\xi=0$ and $\xi=1$.

Frequency equation is

$$\sin\beta = 0 \tag{31}$$

and natural modes are

$$Y=\sin \beta \xi \tag{32}$$

$$\varphi=\cos \beta \xi \tag{33}$$

Case (2) : Hinged at $\xi=0$ and $\xi=1$.

The frequency equation is

$$2-2 \cosh \alpha \cos \beta-\left(\frac{\eta^2-\zeta^2}{\eta\zeta}\right)\sinh \alpha \sin \beta=0 \tag{34}$$

and the natural modes are

$$Y=\cosh \alpha\xi-\cos \beta\xi-\left\{\frac{\cosh \alpha-\cos\beta}{\zeta \sinh \alpha+\eta\sin \beta}\right\}(\zeta \sinh \alpha\xi+\eta \sin \beta\xi) \tag{35}$$

$$\varphi=\cosh \alpha\xi-\cos \beta\xi-\left\{\frac{\cosh \alpha-\cos \beta}{\eta \sinh \alpha-\zeta\sin \beta}\right\}(\eta \sinh \alpha\xi-\zeta \sin \beta\xi) \tag{36}$$

where

$$\eta=\frac{(s^2 (b^2-q)+\alpha^2)}{\alpha} \tag{37}$$

$$\zeta=\frac{\{s^2 (b^2-q)-\beta^2\}}{\beta} \tag{38}$$

Case (3) : Free at $\xi=0$ and $\xi=1$.

The frequency equation, in this case, is

$$2-2 \cosh \alpha \cos \beta-\left\{\frac{\zeta\beta^2}{\eta\alpha^2}-\frac{\eta\alpha^2}{\zeta\beta^2}\right\}\sinh \alpha \sin \beta=0 \tag{39}$$

The natural modes are

$$Y=\zeta\beta \cosh \alpha\xi-\eta\alpha \cos \beta\xi-\left\{\frac{\eta\alpha\, \zeta\beta (\cosh \alpha-\cos \beta)}{\eta\alpha^2 \sinh \alpha-\xi\beta^2 \sin \beta}\right\}\cdot(\alpha \sinh \alpha\xi+\beta \sin \beta\xi) \tag{40}$$

$$\varphi=\eta\alpha \cosh \alpha\xi-\zeta\beta \cos \beta\xi-\left\{\frac{\eta\alpha^2 \sinh \alpha+\zeta\beta^2 \sin \beta}{\alpha\beta (\cosh \alpha-\cos \beta)}\right\}(\beta \sinh \alpha\xi-\alpha \sin \beta\xi) \tag{41}$$

Case (4) : Clamped at $\xi=0$ and hinged at $\xi=1$.

The frequency equation is

$$\frac{\zeta}{\eta} \tanh \alpha+\tan \beta=0 \tag{42}$$

The natural modes are

$$Y=\cosh \alpha\xi-\cos \beta\xi-\left\{\frac{\eta\alpha \cosh \alpha-\zeta\beta \cos \beta}{\eta\zeta(\alpha \sinh \alpha+\beta \sin \beta)}\right\}(\zeta \sinh \nu\zeta+\eta \sin \beta\xi) \tag{43}$$

$$\varphi=\eta \sinh \alpha\xi-\zeta \sin \beta\xi-\left\{\frac{\eta \sinh \alpha-\zeta \sin \beta}{\cosh \alpha-\cos \beta}\right\}(\cosh \alpha\xi-\cos \beta\xi) \tag{44}$$

Case (5) : Clamped at $\xi = 0$ and free at $\xi = 1$.

The frequency equation is

$$\left\{\frac{\alpha\eta}{\beta\zeta}+\frac{\beta\zeta}{\alpha\eta}\right\} \cosh \alpha \cos\beta + \left(\frac{\beta^2 - \alpha^2}{\alpha\beta}\right) \sinh \alpha \sin \beta - 2=0 \tag{45}$$

The natural modes are

$$Y = \cosh \alpha\xi - \cos\beta\zeta - \left\{\frac{\beta \sinh \alpha - \alpha \sin \beta}{\zeta\beta \cosh \alpha - \alpha\eta \cos\beta}\right\} (\zeta \sinh \alpha\xi + \eta \sin \beta\xi) \tag{46}$$

$$\varphi = \cosh \alpha\xi - \cos\beta\xi - \left\{\frac{\alpha \sinh \alpha + \alpha \sin\beta}{\alpha\eta \cosh \alpha - \beta\zeta \cos\beta}\right\} (\eta \sinh \alpha\xi - \zeta \sin \beta\xi) \tag{47}$$

Cose (6) : Hingeg at $\xi=0$ and free at $\xi=1$.

The frequency equation is

$$\frac{\eta\alpha^2}{\zeta\beta^2} \tanh \alpha + \tan \beta=0 \tag{48}$$

The natural modes are

$$Y=\alpha \sinh \alpha\xi + \beta \sin \beta\xi + \left\{\frac{\eta\alpha^2 \sinh \alpha + \zeta\beta^2 \sin\beta}{\eta\zeta\, \alpha\beta (\cosh \alpha - \cos\beta)}\right\} (\zeta\beta \cosh \alpha\xi - \eta\sigma \cos \beta\xi) \tag{49}$$

$$\varphi=\eta\alpha \cosh \alpha\xi - \zeta\beta \cos \beta\xi + \left\{\frac{\eta\alpha^2 \sinh \alpha + \zeta\beta^2 \sin \beta}{\alpha\beta (\cosh \alpha - \cos\beta)}\right\} \beta \sinh \alpha\xi - \alpha \sin \beta) \tag{50}$$

Numerical Example

Inspection of the frequency equations (31), (34), (39), (42), (45) and (48) for the six common combinations of boundary conditions presented above indicates that considerable study and labor is entailed in the solution of each of the equations if roots are to be computed for various combinations of the parameters s, r, and q. Further, the desirability of a graphical presentation of these roots is evident, if the quantitative effects of shear, rotatory inertia, and elastic foundation are to be easily comprehended.

As a particular numerical example, the first four frequencies for a clamped-free beam (Case 5) have been computed from the frequency equation (45). A ratio of $E/G=8/3$ and a shape factor $k = 2/3$ have been assumed. Under these assumptions, $s = 2r$; and the variables α, β, η, and ζ become functions of the rotatory inertia parameter. r, and the foundation parameter q. Accordingly, the frequency equation (45) and its associated roots, b, are functions of these same two parameters, r and q.

In terms of r, q, and b the frequency equation becomes

$$\left\{\frac{4r^2 (b^2-q) + \alpha^2}{4r^2 (b^2-q) + \beta^2} + \frac{4r^2 (b^2-q) - \beta^2}{4r^2 (b^2-q) + \alpha^2}\right\}\cosh \alpha \cos \beta + \left(\frac{\beta^2-\alpha^2}{\alpha\beta}\right) \sinh \alpha \sin \beta - 2 = 0 \tag{51}$$

where α and β are derived from equations (24) and (25) with the condition

$s = 2r$:

$$\alpha = [\tfrac{1}{2} \{r^4 (3b^2-4q)^2 + 4(b^2-q)\}^{1/2} - \tfrac{1}{2} r^2(5b^2-4q)]^{1/2} \tag{52}$$

$$\beta = [\tfrac{1}{2} \{r^4 (3b^2-4q)^2 - 4(b^2-q)\}^{1/2} + \tfrac{1}{2} r^2(5b^2-4q)]^{1/2} \tag{53}$$

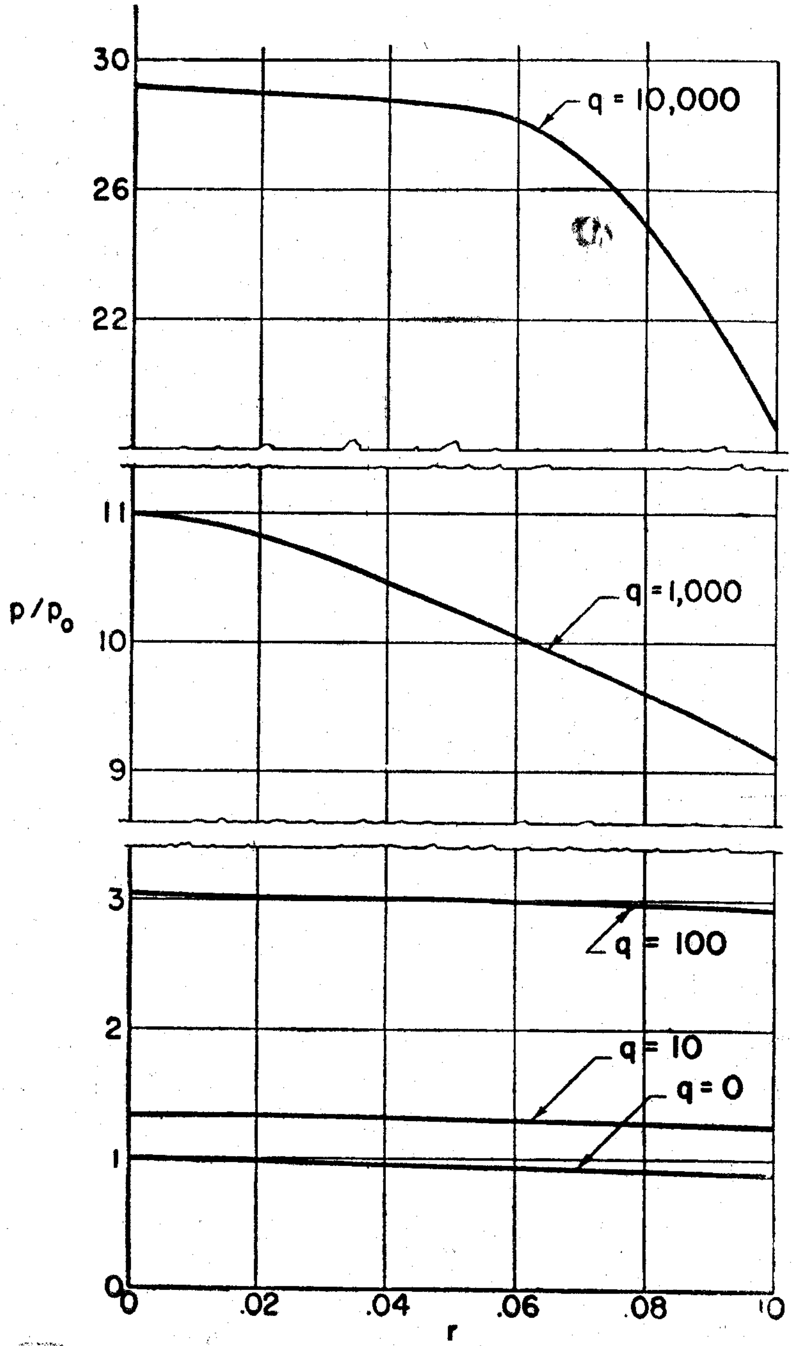
30
26
22
q = 10,000
11
p/p$_o$
10
9
q = 1,000
3
2
1
0
q = 100
q = 10
q = 0
0
.02
.04
.06
.08
.10
r

Fig. 1

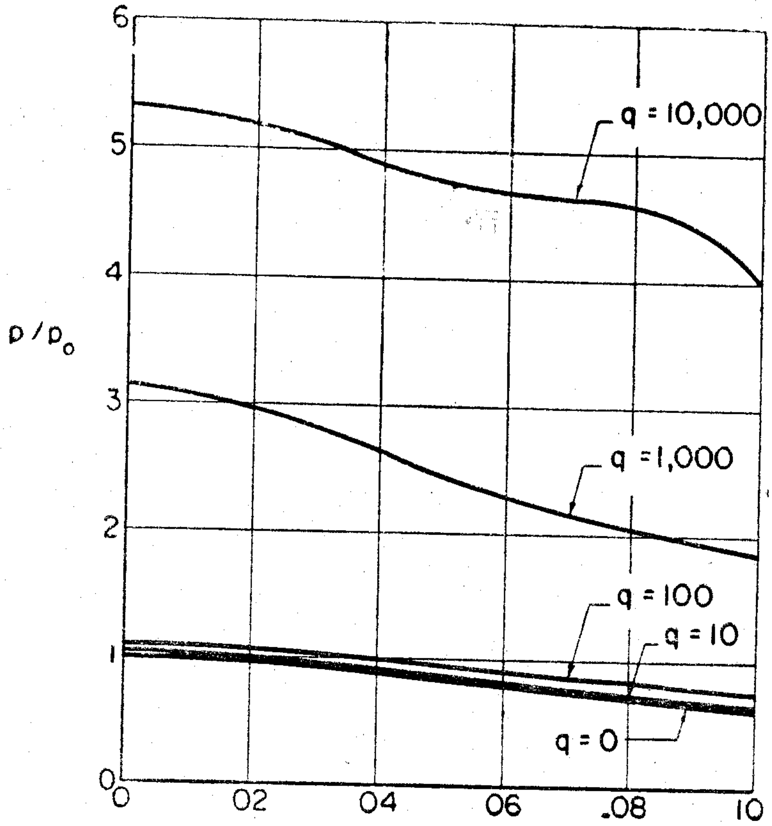

Fig. 2

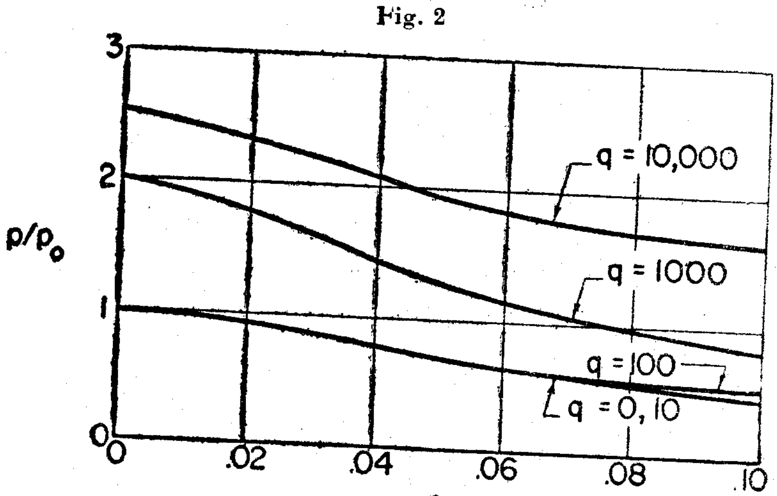

Fig. 3

Equation (51) and the associated equations (52) and (53) form the basis for the solutions presented in figures 1 — 4. As has already been noted, the frequency equation (51) is valid only when a and β are real. It is quite conceivable, however, that a or β could become a pure imaginary function for some values of the parameters r and q. [The case of a and β complex is excluded on the basis of the comments following (26a)]. When a or β is imaginary it is then necessary to make the substitution given in one of the equations (24a) or (25a) in the frequency equation (53). The frequency equation then assumes the following froms:

For a imaginary

$$\left[\frac{4r^2(b^2-q)-a'^2}{4r^2(b^2-q)-\beta^2}+\frac{4r^2(b^2-q)-\beta^2}{4r^2(b^2-q)-a'^2}\right]\cos a'\cos\beta+\frac{\beta^2+a'^2}{a'\beta}\sin a'\sin\beta-2=0 \qquad (54)$$

and for β imaginary

$$\left[\frac{4r^2(b^2-q)+a^2}{4r^2(b^2-q)+\beta'^2}+\frac{4r^2(b^2-q)+\beta'^2}{4r^2(b^2-q)+a^2}\right]\cosh a\cosh\beta' -\left(\frac{a^2+\beta'^2}{a\beta'}\right)\sinh a\sinh\beta'-2=0 \qquad (55)$$

It isinteresting to note that for any one set of the parameters r and q, it is not immediately obvious that there are no roots with β imaginary. For the range of parameters studied, however, no roots were found with β imaginary.

In each of the figures 1 to 4 p_0 is the frequency of a fixed-free beam *in vacuo* from classical theory, the corresponding values of which are

$p_0 = 3.52, 22.03, 61.70, 120.91$

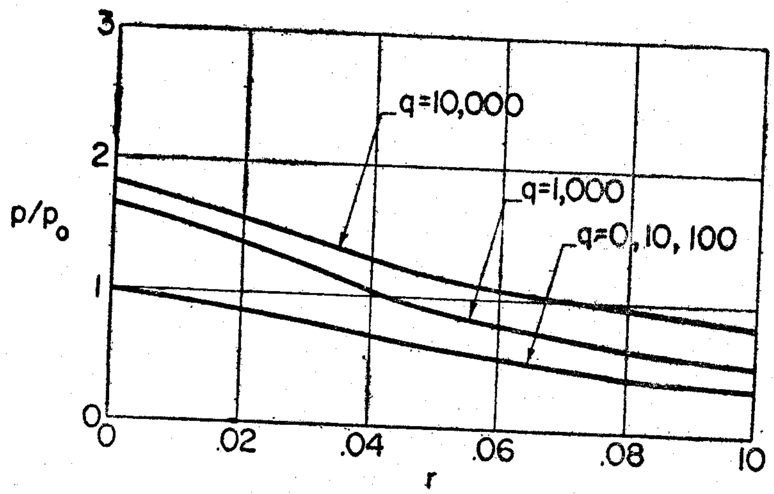

Fig. 4

These figures are the graphical representation of p/p_0 versus r for values of $q = 0, 10, 10^2, 10^3$ and 10^4 for the first four modes of a finite fixed-free beam on an elastic foundation with the range of r from 0 to .10. It is seen that the effect of shear deformation and rotatory inertia increases with the increasing foundation modulus and depth of the beam. Similar conclusion is reached by Tseitlin (15) in his study of beams of infinite length on any elastic foundation. The effect of foundation modulus, of course,decreases, whereas the effect of shear deformation and rotatory inertia increases for the higher modes.

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