Paper No. 25

ON THE APPLICATION OF THE RECIPROCAL THEOREM TO THE VIBRATION OF CONTINUOUS BEAMS

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SYNOPSIS

This paper illustrates the application of reciprocal theorem to the vibration of continuous beams. A few illustrative examples have been given.

INTRODUCTION

The vibration of continuous beams has been extensively studied in the past. The clasical approach for the study of the free vibrations is through the "Three moment" equation or the "slope-deflection" equations. Either of the methods results in a set of homogeneous equations with the support moments as the unknowns. Then the transcendental frequency equation is arrived at by the condition that the determinant formed by the coefficients of the support moments in the set of homogeneous equations must vanish for non-trivial solutions. The steady state forced vibrations of continuous beams has been studied by Gaskell (1952) by a moment balancing procedure analogus to the Hardy Cross method of moment distribution in statics. A serious limitation of this procedure is that the convergence of the solution is not assured for values of the exciting frequency greater than the fundamental natural frequency of the beam.

The steps taken by Saibel and D' Apolonia (1952) for the solution of the forced vibration problems are (a) Intermediate supports are removed leaving an ordinary simple beam for which the eigen values and eigen functions are known, (b) The deflection of the beam at any point is represented as an infinite series in terms of the eigen functions of the simple beam. (c) The constraints at the intermediate supports are introduced through undetermined multipliers. From the Lagrange equations of the motion and the conditions of constraint, equations are developed that yield the solution.

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In this paper a method based on Reciprocal theorem has been developed for the study of vibrations of the continuous beams. The method consists in essentially considering the system in two states, firstly under given loadings; secondly under a convenient fictitious loading and to apply the Reciprocal theorem which states that "if a body is considered, subject to two different states of forces then the work done by the forces ot each state on the corresponding displacements of the other are equal."

The present method gives the same frequency equation as in the classical methods circumventing the formation of the simultaneous equations in the case of free vibrations and in the case of ,forced vibrations it has no problem of convergency, this being not a successive approximation method. Compared to the approach of Saibel and D' Apolonia (1952) the present method for the steady state solution does neither requires a knowledge of the eigen values and eigen functions of the simple beam nor encounters the solution of simultaneous equations. Hence the use of the method lies mostly in the solution of the forced vibration problems of continuous beams.

SOME USEFUL RESULTS

For a simply supported beam subjected to an end moment M Sinpt (Fig. 1) the equation of motion is

$$\frac{\partial^4 y}{\partial x^4}+\frac{m}{EI}\frac{\partial^2 v}{\partial t^2}=0 \tag{1}$$

where m is the mass per unit length and EI is the flexural rigidity of the beam

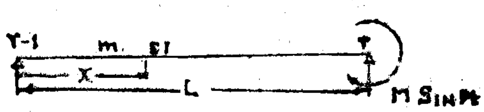

Fig. 1

with the Boundary conditions

$$y=0 \text{ at } x=0 \text{ and } L, \quad \frac{\partial^2 v}{\partial x^2}=0 \text{ at } x=0; \quad \frac{\partial^2 y}{\partial x^2}=M \sin pt \quad \text{at } x=L$$

The solution is

$$y\,(x,\,t)=\frac{M}{2EI\lambda^2}\left[\frac{\sin \lambda x}{\sin \lambda L}-\frac{\sinh \lambda x}{\sinh \lambda L}\right](\sin pt) \tag{2}$$

where $\lambda^4=\frac{mp^2}{EI}$

The slope at the Left and Right ends are respectively

$$\theta_{r-1,\,r}=\frac{M}{2EI\lambda}\left[\frac{1}{\sin\lambda L}-\frac{1}{\sinh\lambda L}\right]\sin pt=\frac{ML}{6EI}\,\psi\sin pt$$

$$\theta_{r,\,r-1}=\frac{M}{2EI\lambda}\left[\cot h\,\lambda L-\cot\lambda L\right]\sin pt=\frac{ML}{3EI}\,\theta\sin pt \tag{3}$$

The dynamic carry over factor

$$=\frac{\theta_{r-1,\,r}}{\theta_{r,\,r-1}}=\frac{\psi}{2\theta} \tag{4}$$

FREE VIBRATION OF CONTINUOUS BEAMS

Referring to Fig. 2 let it be required to find the natural frequencies of the beam on 'n' supports. The condition of the beam freely vibrating, and hence acted on by support moments M_r sinpt etc. is the I state for applying the Reciprocal theorem. The II state is the one in which the beam is acted on by opposite pulsating moments of unit amplitude applied at some support 'r'.

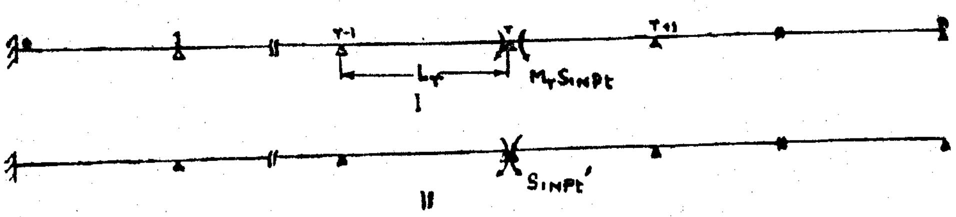

Fig. 2

The deflection of the span (r—1, r) in the I state can be written as

$$y_{rI}=X_{rI}\sin pt$$

and the force acting on the span

$$=\frac{m\partial^2 y_{rI}}{\partial t^2}=-mp^2X_{rI}\sin pt$$

Similarly the deflection in the II state is

$$y_{rII}=X_{rII}\,\text{sinpt}$$

Force acting $\qquad =-mp^2X_{rII}\,\text{sinpt}$

Let $\phi_{r,\,r-1}$ and $\phi_{r,\,r+1}$ be the amplitude of the slopes to the left and right of support 'r' in the II state.

From the Reciprocal Theorem

$$\sum_{r=1}^{n} \int_0^{Lr} m\frac{\partial^2 y_{rI}}{\partial t_2} y_{rII} dx + M_r \text{ Sinpt } (\phi_{r,\ r-1} + \phi_{r,r+1}) \text{ Sinpt}$$

$$= \sum_{r=1}^{n} \int_o^{Lr} m\frac{\partial^2 y_{rII}}{\partial t^2}\ y_{rI}\ d_x \quad (5)$$

This simplifies to

$$\phi_{r,\ r-1} + \phi_{r,\ r+1} = 0 \quad (6)$$

This is the frequancy equation for the beam. The tables of θ and ψ for values of λL within 2π, as given by G.L. Rogers greatly facilitate the solution of equation (6).

Example 1.

To find natural frequencies of the beam shown in fig. 3. m and EI remain constant throughout.

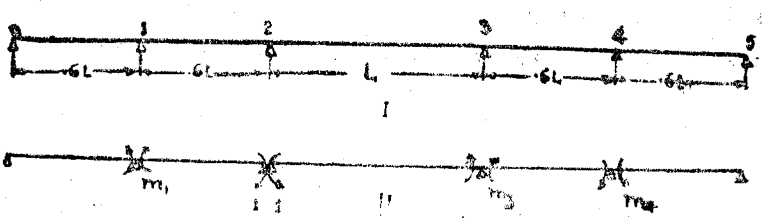

Fig 3

Moments with unit amplitude are applied at support '2' due to which moments of amplitude m_1, m_3 and m_4 are induced at 1, 3 and 4. They are found by successively applying Reciprocal theorem considering the beam to the left and right of support '2' as two separate continuous beams acted on by an end moment.

Henceforth in the figure for the II state only the amplitudes of the different quantities will be shown. Considering the condition of the portion 0–1–2 when acted on by a moment 'Sin pt' at '2' as the I state of loading and the condition in which opposite moments 'Sin pt' act at '1' as the II State from Reciporcal theorem.

$$2m_1 \left(\frac{0.6L}{3EI}\right) \theta_{0\cdot6} = \left(\frac{0.6L}{6EI}\right) \psi_{0\cdot6}$$

$$\text{i.e. } m_1 = \psi_{0\cdot6}/4\theta_{0\cdot6}$$

The subscripts for θ and ψ refer to the ratio of characteristic length (λL_r) of the span under consideration to that of span 2–3.

Next considering the portion 2–3–4–5 (Fig. 4) and applying Reciprocal theorem.

$$m_3 \left[\frac{L\theta}{3EI} + \frac{0.6L}{3EI}\theta_{0\cdot6} - \frac{\psi^2_{0\cdot6}}{4\theta_{0\cdot6}}\left(\frac{0.6L}{6EI}\right)\right] = \frac{L}{6EI}\psi$$

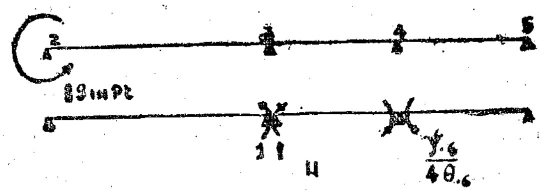

Fig. 4

Hence $m_3 = \left[\dfrac{\psi}{2\theta + 1.2\theta_{0\cdot6} - 0.15\,\dfrac{\psi^2_{0\cdot6}}{\theta_{0\cdot6}}}\right]$

The frequency equation referring to Fig. 3 is

$$\phi_{2,1} + \phi_{2,3} = 0$$

i.e. $\left(\dfrac{0.6L}{3EI}\,\theta_{0\cdot6} - m_1\,\dfrac{0.6L}{6EI}\,\psi_{0\cdot6}\right) + \left(\dfrac{L}{3EI}\,\theta - m_3\,\dfrac{L}{6EI}\,\psi\right) = 0$

This simplifies to

$$0.6\,\theta_{0\cdot6} - 0.075\frac{\psi^2_{0\cdot6}}{\theta_{0\cdot6}} + \theta - \frac{\psi^2}{\left(4\theta + 2.4\theta_{0\cdot6} - 0.3\,\dfrac{\psi^2_{0\cdot6}}{\theta_{0\cdot6}}\right)} = 0$$

The solution of this equation leads to the different natural frequencies. The fundamental frequency is found to correspond to $\lambda L = 3.38$.

FORCED VIBRATION OF CONTINUOUS BEAMS

When pulsating external loads are acting on a continuous beam, the Reciprocal theorem is applied to determine the support moments explicitly. Loads of different excitation frequency are to be considered separately and the effects superposed. Referring to Fig. 5 due to a load W Sinpt on span (r—1, r) oscillatory moments with amplitude Mo, M, etc. are induced at supports 0, 1 etc., considering another condition of the beam in which instead of external loads unit amplitude moments having the same frequency as the load are applied at support 'r'. From Reciprocal theorem

$$-W\delta + M_r\,(\phi_{r,r-1} + \phi_{r,r+1}) = 0 \qquad (7)$$

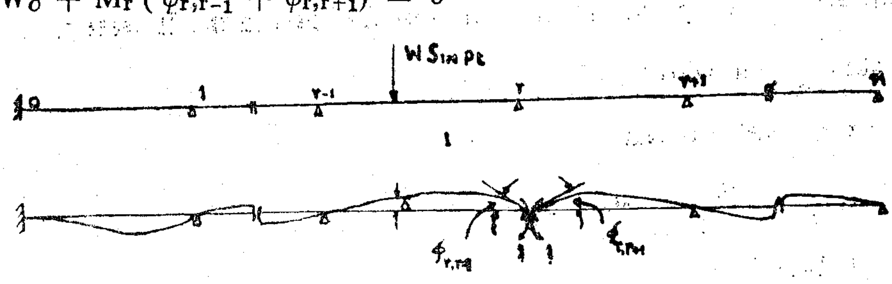

Fig. 5

where δ is the deflection amplitude at the point of application of the load in the II state.

From equation (7) M_r can be found out. The other support moments can be found by considering the beam to the left and right of support 'r' separately, acted on by one end moment M_r along with the given external loads. As the moments induced at the supports due to a moment at support 'r' will have been already calculated in the determination of M_r, the computation of the other support moments will not present much difficulty.

The problems which have been worked by the authors (Gaskell and Saibel & D'Apolonia) mentioned in the introduction are worked here to compare the numerical results.

Example 2.

To find the support moments of the beam shown in Fig. 6. m, EI are constant throughout.

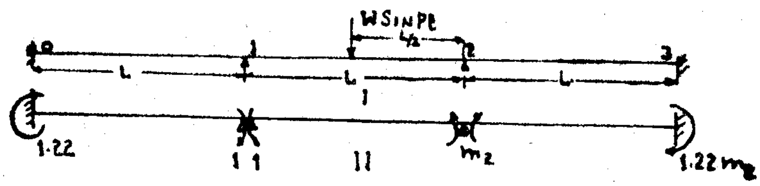

Fig. 6

The frequency of the load is such that $\lambda L = 3.3$.

for $\lambda L = 3.3$. $\theta = -2.3896$ $\psi = -5.8302$

The moment at '0' due to unit moment at '1' in the II state

$$= \frac{\psi}{2\theta} = 1.22$$

Considering the portion 1–2–3 separately in the II state we get

$$m_2 = 2.37$$

$$\phi_{10} = \left(\frac{\theta L}{3EI} - 1.22 \frac{\psi L}{6EI}\right) = 0.3865 \frac{L}{EI}$$

$$\phi_{12} = \left(\frac{\theta L}{3EI} - 2.37 \frac{\psi L}{6EI}\right) = 1.5035 \frac{L}{EI}$$

amplitude of deflection at the point of application of the load in the II state is

$$= 0.405 \frac{L^2}{EI} \text{ upwards}$$

Hence from the Reciprocal theorem

$$M_1 (\phi_{10} + \phi_{12}) - \frac{0.405}{EI} WL^2 = 0$$

i.e. $M_1 = 0.2142$ WL

The sign of M_1 indicates the direction assumed in the beginning for M_1 as shown in Fig. 5 is correct, a -ve sign would have indicated the opposited direction.