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COUPLED FORCED MOTION OF STRUCTURE–FLUID SYSTEMS

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INTRODUCTION

The dynamic characteristics of a structure, which is partially or totally submerged in a fluid medium are substantially different from those due to its vibration in absence of a medium ; the fluid in the Civil Engineering context is usually water, which considerably modifies the response of the structure to a given motion, the extent of modification depending on the characteristics of the structure and the fluid body.

A very important aspect of the problem is the interaction between the structure and the fluid ; the vibrating structure generates Hydrodynamic pressures which modify its deformations ; these deformations in turn modify the Hydrodynamic pressures causing them. The problem is thus recognised as a "coupled" or Elasto-Hydrodynamic one, in which the structure-fluid system may be considered to respond to excitation as a unit, having its own 'coupled' dynamic characteristics, which are different from the uncoupled characteristics of the structure and the fluid separetely. Exactly how different the coupled characteristics would be compared to the uncoupled ones, would depend on the so called "coupling factor" (8) between the structural and fluid phases. For example, for a flexible structure, the effect of coupling would be only to the extent of its so called "added mass" (11, 12), while, for a stiff structure, the characteristics would not be very different from those of the fluid-body itself.

Brahtz and Heilbron (1) were probably the first to study the effect of interaction on Hydrodynamic pressures generated on a vibrating gravity Dam section, by assuming it to deflect linearly during horizontal motions. This assumption however is a rather restricting one for general structure-fluid systems.

The well known "added mass" method has been used extensively (3, 12, 13) to account for the coupling effects. By definition, the "added" or virtual mass of a submerged structure is the mass of fluid that may be imagined to participate in motion, as if it were an additional mass of the structure itself. Tremmel (9) suggested that coupling effects could be represented by adding equal additional masses at the nodes of the structure under consideration. This however was found to be unsatisfactory, as it gave grossly exaggerated results (3). It is more logical however (3) to abandon the assumption of equal nodal added masses and instead assume them to be proportional to the actual Hydrodynamic pressures generated at the interface between the structure and the fluid ; indeed the method can be further refined by coupling the mass matrix of the structure with the full "Hydrodynamic mass matrix" (3), which refers to the interface between the structure and the fluid. This refinement gives very satisfactory results for flexible structures (3, 13). Unfortunately however, this procedure is not suitable for stiff structures ; this is due to the fact that the conventional added mass approach does not recognise the compressibility of the fluid, which now becomes important and has considerable effect on added mass (6).

A satisfactory solution to such problems can be obtained by formulating them as problems of coupled forced vibration (7, 14) ; this recent development is a numerical one in which the stiffness matrix of the structure and the matrix giving the Hydrodynamic pressures are obtained by finite difference or finite element methods. The

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only criticism of this method lies in the fact that whereas the stiffness matrix of the structure can be obtained relatively easily, determination of the Hydrodynamic mass matrix demands a faithful representation of the entire fluid-body by the numerical grid. It is easy to see therefore that with complicated configurations of the structure (which may be curved in space, for example) or the fluid body, one has a real problem ; such problems can of course be surmounted, provided that one has access to large computer stores and sophisticated routines.

This paper aims at giving a brief but comprehensive numerical formulation of the general problem of harmonic coupled forced motion (both horizontal and vertical) of structure-fluid systems. In addition, a simple method of Electric Analogy, developed earlier by the Author (3, 4, 10) is also given. The latter is of particular help in case of complicated configurations of the structure-fluid system.

EQUATIONS OF MOTION

Consider the simple structure-fluid system shown in Fig. 1 ; during horizontal motion, for example, the inertial acceleration, a (t) of the structure is clearly given by,

$$a(t)=a_g(t)+\ddot{z} \qquad \ldots 1$$

where, $a_g(t)$=ground acceleration, z (y, t)=deflections of the structure measured from the y-axis and dots denote differentiation with respect to time. With [S] and [M] as non-dimensional stiffness and mass matrices (note that [ ] and { } respectively denote square matrices and vectors ; $[\ ]^d$ denotes a diagonal matrix) of the structure and ρ as its mass density, its equation of uncoupled damped forced motion can be written in the matrix form, on using Eq. 1 for inertial acceleration, as,

$$\frac{\rho H^2}{E}[M]\{\ddot{z}\}+\frac{c'H}{E}\{\dot{z}\}+[S]\{z\}+\frac{H}{E}\{p'\}=-\frac{\rho H^2}{E}[M]\{a_g\} \qquad \ldots 2$$

in which, E=Young's modulus of the structure, c'=coefficient of equivalent viscous damping and {p'}=Hydrodynamic pressures at the interface between the structure and the fluid.

Assumping the fluid to be frictionless, the Hydrodynamic pressures, p(x, y, t) generated in it during motions of small amplitudes can be shown (10, 12) to be governed by the wave equation,

$$\nabla^2 p=(1/c^2)\ddot{p} \qquad \ldots 3$$

in which, c=acoustic velocity in the fluid and ∇^2 denotes the Laplacian operator in a Cartesian framework. For a finite fluid domain and under steady-state conditions of motion without loss, a standing-wave solution to Eq. 3 is given by the matrix equations,

$$\{p_r\}=q[G]\,[v]^d\,\{p_r\}+[G]\,\{f_r\} \qquad \ldots 4a$$

and

$$\{p_i\}=q[G]\,[v]^d\,\{p_i\}+[G]\,\{f_i\} \qquad \ldots 4b$$

where the suffixes r and i stand for real and imaginary parts of response respectively ; also,

$$\{p\}=\{p_r\}\,\text{Sin}\,(\omega t)+\{p_i\}\,\text{Cos}\,(\omega t) \qquad \ldots 5a$$

$$\{f\}=\{f_r\}\,\text{Sin}\,(\omega t+\{f_i\}\,\text{Cos}\,(\omega t) \qquad \ldots 5b$$

It is to be noted in Eqs. 4 that although the fluid has no damping. {p} is nevertheless complex, since {f} for coupled motion is complex, as will be seen later. Also, $q=(\omega/c)^2$, ω=circular frequency of motion and $[v]^d$ is a weighting matrix of elemental fluid volumes. [G] is the so called "influence matrix" (3) of dynamic pressures and is obtained from the solution of $\nabla^2 p=0$; the vector {f} is obtained from the boundary conditions that {p} must satisfy ; these are :

At the free surface at $y=H_r$, one may assume $p=0$ if waves are not likely to form (5, 10) ; otherwise, a linearised condition such as,

$$(1/g)\ddot{p}+\frac{\partial p}{\partial y}=0 \qquad \text{at } y=H_r \qquad \ldots 6$$

where, g=gravity acceleration, may be assumed (5, 10). At moving boundaries one must have (3, 12, 14), with m=mass density of water,

$$\frac{\partial p}{\partial n}+ma'(t)=0 \qquad \ldots 7$$

where, n is the direction along the normal to the moving surface ; similarly (5, 10), at stationary boundaries one must have,

$$\frac{\partial p}{\partial n}=0 \qquad \ldots 8$$

Clearly, {p} in Eqs. 4 gives dynamic pressures at all the pivots within the fluid ; however, for the solution of Eq. 2 only the amplitudes of {p'}, given by,

$$\{p'\}=\{p_r\}\ \text{Sin}\ (\omega t)+\{p_i\}\ \text{Cos}\ (\omega t) \qquad \ldots 9$$

at the interface are required (note that {p'} is a sub-vector of {p} for the interface) ; {p'} can be extracted from {p} by suitably partitioning Eqs. 4. so that, with [D] as a sub-matrix of [G], referring only to the interface, one can write on assuming equal grid spacing for simplicity,

$$\{p_r\}=q[D]\ \{p_r\}+[D]\ \{f'_r\} \qquad \ldots 10a$$

and

$$\{p_i\}=q[D]\ \{p_i\}+[D]\ \{f'_i\} \qquad \ldots 10b$$

in which {f'} is the interfacial sub-vector of {f}, given by,

$$\{f'\}=\{f'_r\}\ \text{Sin}\ (\omega t)+\{f'_i\}\ \text{Cos}\ (\omega t) \qquad \ldots 11$$

With $p=0$ at $y=H_r$ for simplicity, non-zero conditions are found to exist only at the interface ; consequently (1), it can be shown from Eqs. 1 and 7 that,

$$\{f'\}=mH_r\ \{a'\ (t)\}=mH_r\ \{a\ (t)+\ddot{z}\} \qquad \ldots 12$$

with

$$\{a_g\ (t)\}=\{a\}\ \text{Sin}\ (\omega t)$$

and

$$\{z\}=\{z_r\}\ \text{Sin}\ (\omega t)+\{Z_i\}\ \text{Cos}\ (\omega t) \qquad \ldots 13$$

one obtains from Eqs. 11 and 12,

$$\{f'_r\}=mH_r\ \{a-\omega^2 z_r\} \qquad \ldots 14a$$

and

$$\{f'_i\}=-mH_r\omega^2\ \{z_i\} \qquad \ldots 14b$$

Now, substituting Eqs. 14 into Eqs. 10, andwith [I] as the unit matrix, one has,

$$\{P_r\}=mH_r\ [KD]\ \{a\}-mH_r\omega^2\ [KD]\{z_r\} \qquad \ldots 15a$$

and

$$\{P_i\}=-mH_r\omega^2\ [KD]\{z_i\} \qquad \ldots 15b$$

in which,

$$[K]=[I-qD]^{-1} \qquad \ldots 16$$

Now, using Eqs. 9, 13 and 15, Eq. 2 can be written in the compact matrix form, after simplification, as,

$$\left[\begin{array}{c|c} \left[S-\lambda\left(M+\frac{r}{k}KD\right)\right] & -\left[\frac{c'H\omega}{E}\right]^d \\ \hline \left[\frac{c'H\omega}{E}\right]^d & \left[S-\lambda\left(M+\frac{r}{k}KD\right)\right] \end{array}\right]\left\{\begin{array}{c} z_r \\ \hline z_i \end{array}\right\} = \left\{\begin{array}{c} -\frac{\rho H^2}{E}\left[M+\frac{r}{k}KD\right]\{a\} \\ \hline O \end{array}\right\} \qquad \ldots 17$$

in which, $r=H_r/H$, $k=\rho/m$ and $\lambda=(\rho H^2\omega^2/E)$; solution of Eq. 17 gives the deflection response of the structure during horizontal motion of the structure-fluid system.

THE ANALOGY

An indicated earlier, [D] is the interfacial sub-matrix of [G], which in turn is the solution of,

$$\nabla^2 p=0 \quad \ldots 18$$

Matrix [D] can therefore be defined as the "influence matrix" of incompressible pressures at the interface. If numerical methods are used to solve Eq. 17, [D] can be obtained only by processing the complete [G] matrix, which is usually large; however, a simple electric analogy method described below enables one to determine [D], without processing the large [G] matrix.

This analogy method has been described in detail in references 6 and 11 ; however, a reasonable account of it is given here.

It is known that in a homogeneous electric conducting medium without sources or sinks, the electric potential function, V(x, y) obeys the equation,

$$\nabla^2 V=0 \quad \ldots 19$$

The similarity between Eqs. 18 and 19 forms the basis of the analogy. Further, as Eq. 18 represents an incompressible solution, Eq. 7 now simply reads,

$$\frac{\partial p}{\partial n}+ma=0 \quad \ldots 20$$

where, a is any acceleration ; the electric current intensity, i_n and the potential function V are also related through a relation similar to Eq. 20. namely,

$$\sigma i_n+\frac{\partial V}{\partial n}=0 \quad \ldots 21$$

in which, σ=specific electrical resistance of the medium.

The scale factor between p and V can now be established from Eqs. 20 and 21 ; for example, with

$$i_n=ma/\sigma \quad \ldots 22$$

the hydrodynamic pressures will be exactly equal to the electric potentials.

In order to obtain [D] for the interface, the structure-fluid system is now replaced by the analogue model, as shown in Fig. 2; as p=0 (*i.e.* V=0 in the analogue) at $y=H_r$, this surface can be represented by a heavy conductor, with respect to which all other potentials will be measured. Eq. 8 requires that the gradient of pressure at stationary boundaries vanish, *i.e.*, as seen from Eq. 21, the current intensities at these boundaries in the analogue are zero ; this can be achieved by modelling the stationary boundaries of the analogue with insulators, perspex for example. Finally, Eqs. 20 and 21 indicate that the moving surfaces must be fed with current intensities given by Eq. 22 ; simultaneous feeding of these currents presents practical difficulties, which can be overcome by determining the so called "influence matrix" of voltages for the moving surface. By definition, an element V_{ij} of the influence matrix [V] represents the voltage produced at the i-th. node due to unit current at the j-th. node. [V] is symmetric owing to the reciprocity of currents and voltages. The [V] matrix can now be obtained by first dividing the entire moving surface into a number of grid areas ; each of these areas is now fed with unit currents through finite electrodes (the dimensions of which are half the grid dimensions (3, 10), while the consequent voltages at all the electrodes are measured in turn with a precision voltmeter. These voltages are in fact elements of the [V] matrix. The simple circuit required to obtain [V] is shown in Fig. 2. Let [W] be a weighting matrix of the grid areas at the interface ; then, it is easy to see that for the interface,

$$\{p\}=(m/\sigma)\ [V]\ [W]^{d}\ \{a\} \qquad ...23$$

Eq. 23 obviously gives the incompressible dynamic pressures at the interface. Also, one has from Eq. 20 and the incompressible (*i.e.*, $q=0$) form of Eq. 10a,

$$\{p\}=[D]\ \{ma\} \qquad ...24$$

It is clear from Eqs. 23 and 24 that,

$$[D]=(1/\sigma)\ [V]\ [W]^{d} \qquad ...25$$

Thus, the [D] matrix can be obtained once [V] has been found and σ determined by using standard equipment.

AN EXAMPLE

The response of a structure-fluid system due to horizontal motion is considerably greater than that due to vertical motion (7) ; the response of the system of Fig. 1 was therefore evaluated for horizontal motion. The following ratios were assumed :

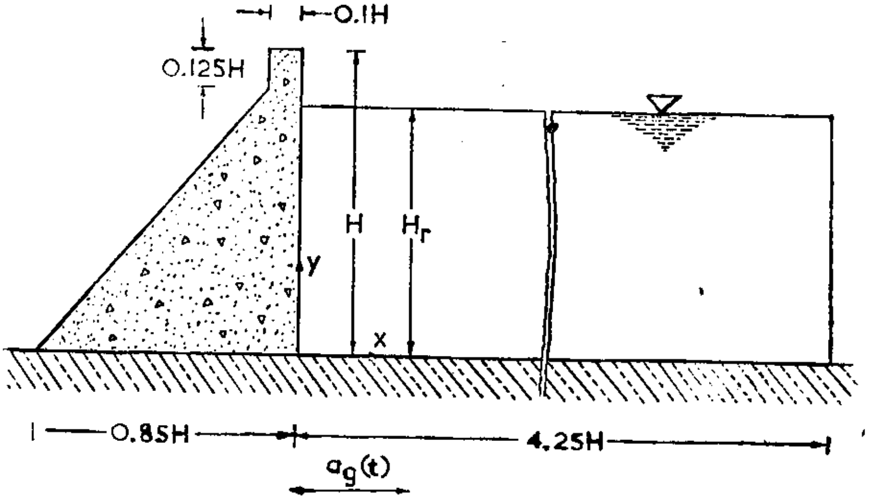

Fig 1. A simple structure-fluid system.

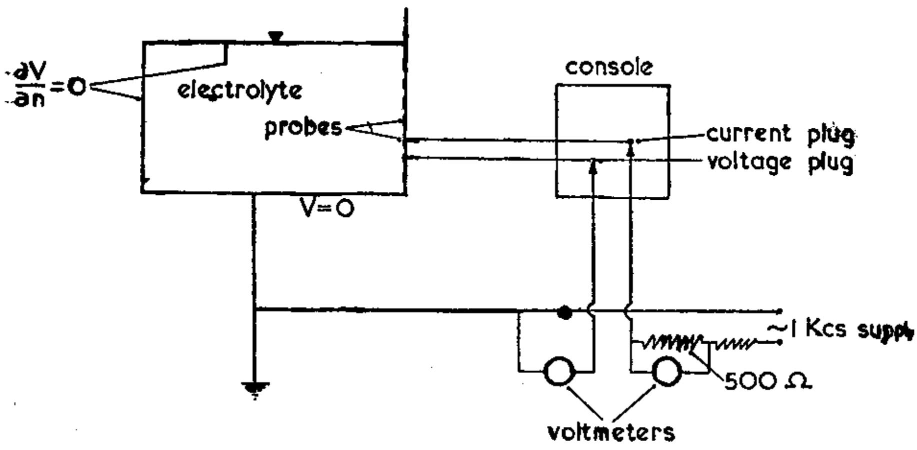

Fig. 2. Analogue scheme for horizontal motion.

$\rho/m = 2{\cdot}4$, E/Bulk modulus of water $= 13{\cdot}33$, Poisson's ratio of the Dam $= 0{\cdot}2$ and $r = 1$ (reservoir full condition).

The [D] matrix was obtained by analogy, by providing the interface (Fig. 2) with 7 equal (spacing $= H_f/8$) finite electrodes (probes).

It is seen from Eq. 17 that its solution also requires the determination of the [K] matrix; Eqs. 15 and 16 show howeuer, that [K] in fact represents the magnification of uncoupled dynamic pressures on the Dam. Considering the fundamental mode first, dynamic magnification (DMF) of uncoupled pressures for this mode can be expressed by the approximate function, Ψ, given by,

$$\text{DMF} \equiv \Psi = 1/(1-\{n_f/n\}^2) \qquad \ldots 26$$

in which, $n_f = cT_f/H_f$ and $n = cT/H$: T and T_f are respectively period of forced motion and fundamental natural period of the fluid-body. n_f at various modes can be obtained from (2),

$$n_f = 4/(2j-1),\ j = 1, 2, 3. \qquad \ldots 27$$

Equation 27, which is valid for the fluid body of Fig. 1, gives $n_f = 4$ for the fundamental mode; using this value in Eq. 26, the approximate DMF given by Ψ is found to agree

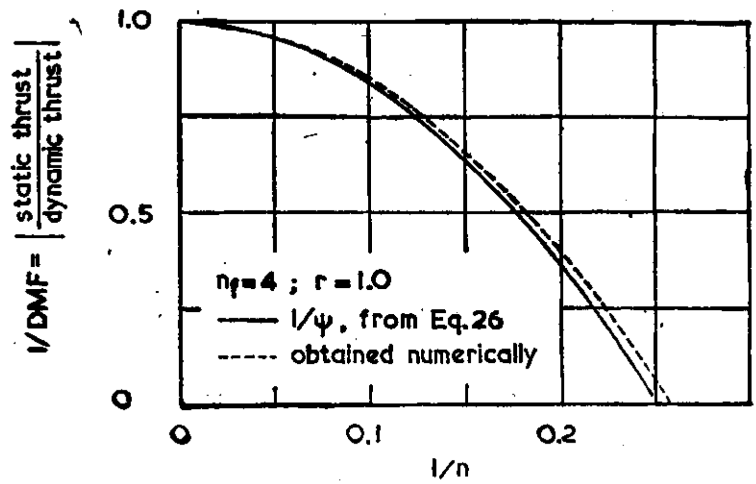

Fig. 3. Assumed and actual dynamic magnifications.

very well with its more accurate variation determined numerically, as seen from Fig. 3. The function Ψ can therefore be substituted for [K] in Eq. 17, which can now be processed for response in the fundamental mode.

The coupled fundamental natural frequency of the system was found to be at $n = 4.46$; Fig. 4 shows the coupled deflections of the Dam at some sub-fundamental frequencies and at 20% critical damping, compared to the corresponding deflections obtained by the numerical processing of Eq. 17; the agreement is seen to be very good.

Response of the system at frequencies higher than the fundamental is relatively unimportant from the practical point of view; however, if required, the analogy method can also be used in this instance by defining Ψ for the first harmonic mode, for example, as,

$$\Psi = 1/\{1-(n_f/n)^2\}\ \{1-(n_{f_1}/n)^2\} \qquad \ldots 28$$

in which, n_{f_1} is the value of n at the first harmonic resonance, to be determined from Eq. 27; the deflection response at such a frequency is shown in Fig. 4, compared to the corresponding numerical plot. Here again, the agreement is good.

UNCOUPLED FUNDAMENTAL FREQUENCY OF THE FLUID-BODY

For most practical situations, determination of response in the fundamental mode is most important. If the method of electric analogy is used, the uncoupled fundamental natural frequency of the fluid-body must be known, so that [K] can be approximated. Exact theoretical values of it can be derived for the simple case considered in this paper; however, determination of this quantity for complicated configurations presents considerable difficulties. Fig. 5 gives the uncoupled fundamental natural frequencies of the fluid-body for three different shapes of reservoir-Dam valley sections commonly met in Civil Engineering practice. These were determined numerically by assuming that the Dam could be approximated by a plate structure; this assumption is

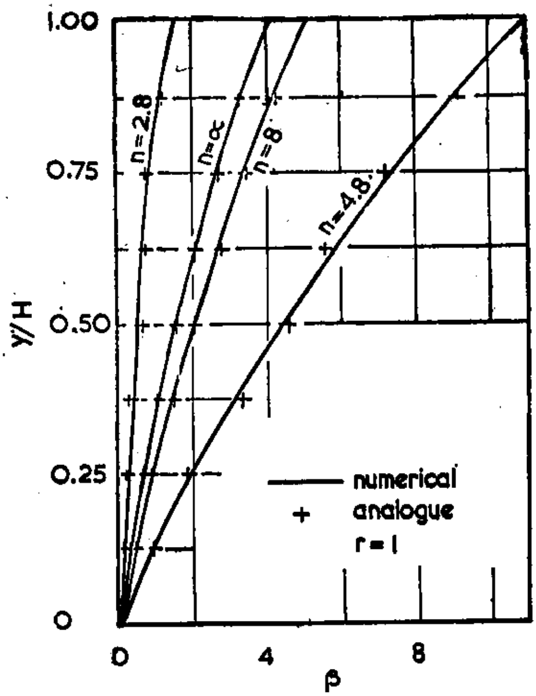

Fig. 4. Coupled deflection at 20% critical damping ; $z=\rho\beta aH^2/E=(z_r^2+z_i^2)^{1/2}$.

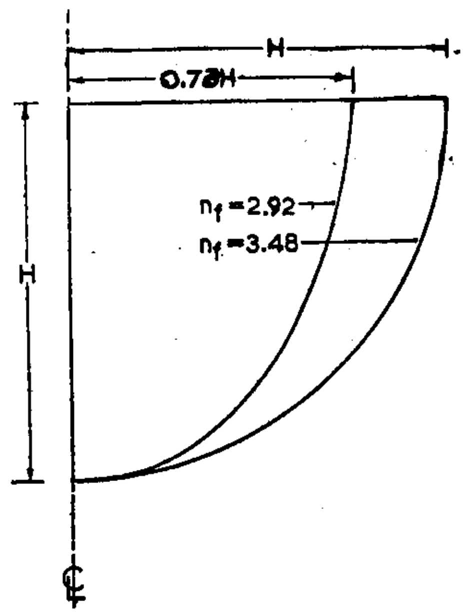

Fig. 5. Values of n_f for the fluid body for some valley shapes.

reasonable, as the effect of mild curvature of the Dam on the fundamental natural frequency of the reservoir was found to be small.

CONCLUSIONS

The response of a structure-fluid system due to vertical motion is of little practical importance (7), compared to its horizontal motion. The latter can be satisfactorily evaluated by using the simple and compact numerical formulation presented in this paper. Further, the alternative simple analogy method discussed here can be used with advantage in cases of complicated configurations. Finally, once the deflection responses are known, stresses in the structure can be easily calculated.

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