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GENERATION OF DISPLACEMENTS AND STRESSES IN A MULTILAYERED HALF-SPACE DUE TO STRIP-LOADING

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ABSTRACT

The problem of the static deformation of a multilayered half-space due to strip loading is studied. The transfer matrix approach is used to find the deformation field at any point of the medium. Both plane strain and antiplane strain cases are discussed in detail. Explicit expression for the displacements and stresses in a uniform half-space and a layer over a half-space are derived. The results for a normal line load and a shear line load are obtained as a particular case. The present formulation is simple and is quite convenient for numerical computation.

1. INTRODUCTION

The behaviour of horizontally multilayered elastic materials under the action of surface loads is of great interest in geophysics, soil mechanics and engineering. Many earthworks, such as fills or pavements, consists of horizontal layers of materials of different kinds. Kuo (1969) studied the three-dimensional problem of an inclined static load on a circular area of the surface of a multilayered isotropic half-space. Singh (1970) solved the corresponding problem for three-dimensional buried sources.

The two-dimensional plane strain and antiplane strain problems of the static deformation of a multilayered isotropic half-space by a normal line load and a shear line load have been discussed by Garg and Singh (1985). The corresponding two-dimensional problem of a long displacement dislocation in a multilayered half-space has been solved by Singh and Garg (1985). Singh (1986), Garg and Singh (1987), Pan (1989) have also studied the deformation of a semi-infinite medium by surface loads.

In the present paper, we have formulated the two-dimensional problem of a normal and shear static strip-loading acting on the boundary of a multilayered isotropic half-space. Both plane strain and antiplane strain problems are solved. It is shown that the integrals giving the stresses can be evaluated analytically when the medium is a uniform half-space. As a particular case, the antiplane strain problem of the deformation of an isotropic medium consisting of a single layer lying over a half-space by the shear strip-loading is solved.

The present paper may be considered as an improvement of the paper by Garg and Singh (1985) in the sense that the results of Garg and Singh (1985) have been derived in the present paper as a particular case. We have used Fourier exponential transformation as the range of the variable y on the boundary of the multilayered medium is $(-\infty, \infty)$. It may also find applications in the study of reservoir-induced seismicity in addition to other fields mentioned earlier. The technique developed in the present paper is simple and straightforward.

2. BASIC EQUATIONS

The equilibrium equations for zero body forces in cartesian system (x_1, x_2, x_3) are

$$\frac{\partial p_{11}}{\partial x_1} + \frac{\partial p_{12}}{\partial x_2} + \frac{\partial p_{13}}{\partial x_3} = 0 \quad (2.1)$$

$$\frac{\partial p_{21}}{\partial x_1} + \frac{\partial p_{22}}{\partial x_2} + \frac{\partial p_{23}}{\partial x_3} = 0 \quad (2.2)$$

$$\frac{\partial p_{31}}{\partial x_1} + \frac{\partial p_{32}}{\partial x_2} + \frac{\partial p_{33}}{\partial x_3} = 0 \quad (2.3)$$

where p_{ij} is the stress tensor. The stress-strain relations are

$$p_{ij} = \lambda \theta \delta_{ij} + 2 \mu e_{ij} \quad (2.4)$$

where e_{ij} is the strain tensor, λ and μ are Lame's constants and θ is the cubical dilatation given by

$$\theta = e_{11} + e_{22} + e_{33} \quad (2.5)$$

If (u_1, u_2, u_3) denote the displacement components, the strain-displacement relations are –

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad (i, j = 1,2,3) \quad (2.6)$$

We shall be considering a two-dimensional approximation in which the displacement components and consequently stresses are independent of x_1 so that $(\partial / \partial x_1) = 0$. Under this assumption, the plain strain problem and the antiplane strain problem are decoupled and, therefore, can be treated separately. In the following, we shall write (x, y, z) for (x_1, x_2, x_3) and (u, v, w) for (u_1, u_2, u_3).

3. ANTIPLANE STRAIN PROBLEM

For the antiplane strain problem, the displacement components are of the type

$$u = u(y,z), \qquad v = w = 0 \tag{3.1}$$

and the non-zero stresses are

$$p_{12} = \mu \frac{\partial u}{\partial y} \qquad p_{13} = \mu \frac{\partial u}{\partial z} \tag{3.2}$$

We notice that the equilibrium equations (2.2) and (2.3) are identically satisfied and (2.1) reduces to

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{3.3}$$

To solve (3.3), we shall make use of Fourier transform. The Fourier transform of $X(y,z)$ is defined as

$$\overline{X}(k, z) = \int_{-\infty}^{\infty} X(y, z)\, e^{iky}\, dy \tag{3.4}$$

so that

$$X(y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X}(k, z)\, e^{-iky}\, dk \tag{3.5}$$

Taking the Fourier transform of (3.3), we obtain

$$\left(\frac{d^2}{dz^2} - k^2\right) \overline{u} = 0 \tag{3.6}$$

Solution of (3.6) is

$$\overline{u} = E\, e^{-|k|\, z} + F\, e^{|k|\, z} \tag{3.7}$$

where E and F may depend upon k.

Hence

$$u = \frac{1}{2\pi} \int_{-\infty}^{\infty} (E\, e^{-|k|\, z} + F\, e^{|k|\, z})\, e^{-iky}\, dk \tag{3.8}$$

Equation (3.2) and (3.8) give

$$p_{13} = \frac{\mu}{2\pi} \int_{-\infty}^{\infty} (-E\, e^{-|k|z} + F\, e^{|k|z})\, e^{-iky} |k|\, dk \tag{3.9}$$

Equation (3.8) and (3.9) may be written as

$$u = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(z)\; e^{-iky}\, dk \tag{3.10}$$

$$p_{13} = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(z)\; e^{-ky}\, |k|\, dk \tag{3.11}$$

The functions U(z) and T(z) are given by the matrix relation

$$[Y(z)] = [Z(z)]\,[K] \tag{3.12}$$

where

$$[Y(z)] = [U(z),\ T(z)]^T, \quad [K] = [E + F,\ E - F]^T \tag{3.13}$$

and $[----]^T$ denotes the transpose of the matrix [------]. The matrix [Z(z)] is given below :

$$[Z(z)] = \begin{bmatrix} ch\,\phi & - & sh\phi \\ \mu\, sh\phi & - & \mu\, ch\,\phi \end{bmatrix} \tag{3.14}$$

where $\phi = |k|\, z$, (3.15)

and ch and sh stand for hyperbolic cosine and sine, respectively.

4. PLANE STRAIN PROBLEM

For the plane strain problem, the displacement components are of the type

$$u = 0, \quad v = v(y,z), \quad w = w(y,z) \tag{4.1}$$

and the non-zero stresses are

$$p_{11} = \mu\left(\frac{1-2\alpha}{\alpha - 1}\right)\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{4.2}$$

$$p_{22} = \frac{\mu}{\alpha-1}\left[-\frac{\partial v}{\partial y} + (1-2\alpha)\frac{\partial w}{\partial z}\right] \tag{4.3}$$

$$p_{33} = \frac{\mu}{\alpha-1}\left[(1-2\alpha)\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right] \tag{4.4}$$

$$p_{23} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \tag{4.5}$$

where

$$\alpha = \frac{\lambda + \mu}{\lambda + 2\mu} \tag{4.6}$$

The equilibrium equation (2.1) is identically satisfied and (2.2) and (2.3) reduce to

$$\frac{\partial p_{22}}{\partial y} + \frac{\partial p_{23}}{\partial z} = 0 \tag{4.7a}$$

$$\frac{\partial p_{23}}{\partial y} + \frac{\partial p_{33}}{\partial z} = 0 \tag{4.7b}$$

Therefore, there exists an Airy stress function $G(y,z)$ such that

$$p_{22} = \frac{\partial^2 G}{\partial z^2}, \qquad p_{23} = \frac{-\partial^2 G}{\partial y \, \partial z}, \qquad p_{33} = \frac{\partial^2 G}{\partial y^2} \tag{4.8}$$

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)^2 G = 0 \tag{4.9}$$

We note that the equilibrium equations (4.7 a, b) are identically satisfied. Fourier transform of (4.9) yields

$$\left(\frac{d^2}{d z^2} - k^2 \right)^2 \bar{G} = 0 \tag{4.10}$$

General solution of (4.10) is

$$\bar{G} = A e^{-|k|z} + B e^{|k|z} + |k|z \left(C e^{-|k|z} + D e^{|k|z} \right) \tag{4.11}$$

where, A, B, C and D may be functions of K.

Inverting (4.11) by means of the Fourier inversion theorem, we obtain

$$G = \frac{1}{2\pi} \int_{-\infty}^{\infty} [A e^{-|k|z} + B e^{|k|z} + |k|z \left(C e^{-|k|z} + D e^{|k|z} \right)] e^{-iky} dk. \qquad \ldots (4.12)$$

From (4.8) and (4.12), the stresses are found to be

$$p_{22} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [A\, e^{-|k|z} + B\, e^{|k|z} - C\,(2 - |k|z)\, e^{-|k|z}$$

$$+ D\,(2 + |k|z)\, e^{|k|z}]\ e^{-iky}\ K^2\, dk \qquad ......(4.13)$$

$$p_{23} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [Ae^{-|k|z} - Be^{|k|z} - C\,(1 - |k|z)\, e^{-|k|z}$$

$$- D\,(1 + |k|z) e^{|k|z}]\ e^{-iky} |k| k\ dk \qquad .....(4.14)$$

$$p_{33} = \frac{-1}{2\pi} \int_{-\infty}^{\infty} [Ae^{-|k|z} + Be^{|k|z} + |k|z\ (Ce^{-|k|z} + De^{|k|z})]$$

$$e^{-iky}\ K^2\ dk \qquad ....(4.15)$$

The formulae for displacements can be obtained by integrating the stress-displacement relations. Using (4.8) and neglecting rigid body displacements, it can be shown that (Sokolnikoff, 1956; section 71).

$$2\mu v = -\frac{\partial G}{\partial y} + \frac{1}{2\alpha} \int (p_{22} + p_{33})\ dy \qquad (4.16)$$

$$2\mu w = -\frac{\partial G}{\partial z} + \frac{1}{2} \int (p_{22} + p_{33})\ dz \qquad (4.17)$$

Equations (4.12), (4.13), (4.15) - (4.17) yield

$$2\mu v = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} [Ae^{-|k|z} + Be^{|k|z} - C(\frac{1}{\alpha} - |k|z) e^{-|k|z}$$

$$+ D\,(\frac{1}{\alpha} + |k|z)\ e^{|k|z}]\ e^{-iky}\ k\ dk \qquad (4.19)$$

$$2\mu w = \frac{1}{2\pi} \int_{-\infty}^{\infty} [Ae^{-|k|z} - Be^{|k|z} + C\,(\frac{1}{\alpha} - 1 + |k|z)\, e^{-|k|z}$$

$$+ D\,(\frac{1}{\alpha} - 1 - |k|z)\ e^{|k|z}]\ e^{-iky} |k|\ dk \qquad (4.20)$$

Equations (4.14), (4.15), (4.19) and (4.20) may be written as

$$v = \frac{1}{2\pi i}\int_{-\infty}^{\infty} V\, e^{-iky}\, k\, dk \tag{4.21}$$

$$w = \frac{1}{2\pi}\int_{-\infty}^{\infty} W\, e^{-iky}|k|\, dk \tag{4.22}$$

$$p_{23} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} S\, e^{-iky}\, |k|k\, dk \tag{4.23}$$

$$p_{33} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} N\, e^{-iky}\, k^2\, dk \tag{4.24}$$

The function V, W, S, N are given by the matrix relation

$$[Y(z)] = [Z(z)]\,[K] \tag{4.25}$$

where

$$[Y(z)] = [V, W, S, N]^T, \qquad [K] = [A+B, A-B, C+D, C-D]^T \tag{4.26}$$

The matrix [Z(z)] is given below :

$$[Z(z)] = \begin{bmatrix} \frac{-\mathrm{ch}\phi}{2\mu} & \frac{\mathrm{sh}\phi}{2\mu} & \frac{-(\mathrm{sh}\phi + \alpha\phi\,\mathrm{ch}\phi)}{2\mu\alpha} & \frac{\mathrm{ch}\phi + \alpha\phi\,\mathrm{sh}\phi}{2\mu\alpha} \\ \frac{-\mathrm{sh}\phi}{2\mu} & \frac{\mathrm{ch}\phi}{2\mu} & \frac{-[\mathrm{ch}\phi + \phi\,\mathrm{sh}\phi - (1/\alpha)\,\mathrm{ch}\phi)}{2\mu} & \frac{\mathrm{sh}\phi + \phi\,\mathrm{ch}\phi - (1/\alpha)\,\mathrm{sh}\phi}{2\mu} \\ -\mathrm{sh}\phi & \mathrm{ch}\phi & -(\mathrm{ch}\phi + \phi\,\mathrm{sh}\phi) & \mathrm{sh}\phi + \phi\,\mathrm{ch}\phi \\ -\mathrm{ch}\phi & \mathrm{sh}\phi & -\phi\,\mathrm{ch}\phi & \phi\,\mathrm{sh}\phi \end{bmatrix} \tag{4.27}$$

5. DEFORMATION OF A MULTISTOREYED HALF-SPACE BY SURFACE LOADS

We consider a semi-infinite elastic medium made up of p-1 parallel homogeneous isotropic layers lying over a homogeneous isotropic elastic half-space. The layers are numbered serially, the layer at the top being layer 1 and the half-space is termed as layer p. We place the origin of the cartesian coordinate system (x, y, z) at the boundary of the semi-infinite medium and the z-axis is drawn into the medium. The mth layer is of thickness d_m and is bounded by the interfaces $z = z_{m-1}$, z_m so that $dm = z_m - z_{m-1}$. Lame's parameters of the mth layer are λ_m, μ_m. Obviously $z_o = 0$ and $z_{p-1} = H$, H being the depth of the last interface (Fig. 1).

Introducing the subscript 'm' to the quantities related to the mth layer, equation (3.12) for the antiplane strain problem and equation (4.25) for the plane strain problem can both be written in the form

$$[Y_m(z)] = [Z_m(z)]\,[K_m], \quad z_{m-1} \leq z \leq z_m \tag{5.1}$$

We assume that the layers are in welded contact. Using the continuity of the displacements and stresses across the interface $z = z_{m-1}$, it has been shown by Singh (1970) and Singh and Garg (1985) that the deformation fields at the boundaries of the consecutive layers satisfy the relation

$$[Y_{m-1}(z_{m-1})] = [a_m]\,[Y_m(z_m)] \tag{5.2}$$

where the transfer matrix $[a_m]$ is given by

$$[a_m] = [Z_m(z_{m-1})]\,[Z_m(z_m)]^{-1} = [Z_m(-d_m)]\,[Z_m(0)]^{-1} \tag{5.3}$$

The elements of the matrix $[a_m]$ are given in the Appendix-I.

Making a repeated use of (5.1) and (5.2), we find

$$[Y_1(C)] = [M]\,[K_p] \tag{5.4}$$

where

$$[M] = [a_1]\,[a_2] \text{ ------ } [a_{p-1}]\,[Z_p(H)] \tag{5.5}$$

The field at any point of the mth layer is obtained from the relation

$$[Y_m(z)] = [N_m]\,[K_p], \quad z_{m-1} \leq z \leq z_m \tag{5.6}$$

where

$$[N_m] = [a_m(z_m - z)]\,[a_{m+1}] \text{ ----- } [a_{p-1}]\,[Z_p(H)] \tag{5.7}$$

and the matrix $[a_m(z_m - z)]$ is obtained from $[a_m]$ on replacing d_m by $z_m - z$.

5.1 Antiplane Strain

For the half-space, $F_p = 0$, since, otherwise, $u \to \infty$ as $z \to \infty$. Equation (5.4) now gives

$$T(0) = (M_{21} + M_{22})\,E_p \tag{5.8}$$

When the surface load is prescribed, E_p can be determined from (5.8). Let the boundary condition be of the type

$$p_{13} = f(y) \quad \text{at} \quad z = 0 \tag{5.9}$$

We write

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(k)\, e^{-iky}\, dk \tag{5.10}$$

where

$$\bar{f}(k) = \int_{-\infty}^{\infty} f(y)\, e^{iky}\, dy \tag{5.11}$$

Equation (3.11) and (5.8) - (5.10) yield

$$E_p = \frac{\bar{f}(k)}{|k|\,(M_{21} + M_{22})} \tag{5.12}$$

The displacement u at any point of the mth layer is obtained from (3.10), (5.6) and (5.12). We find

$$u = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\bar{f}(k)}{|k|} \left(\frac{N_{11} + N_{12}}{M_{21} + M_{22}}\right) e^{-iky}\, dk \tag{5.13}$$

The stresses p_{12} and p_{13} can be obtained from (3.2) and (5.13).

5.2 Plane Strain

Taking $B_p = D_p = 0$, (5.4) yields

$$S(0) = (M_{31} + M_{32})\, A_p + (M_{33} + M_{34})\, C_p \tag{5.14}$$

$$N(0) = (M_{41} + M_{42})\, A_p + (M_{43} + M_{44})\, C_p \tag{5.15}$$

For given surface loads, A_p and C_p are known from (5.14) and (5.15). **Let the** boundary conditions be of the form

$$p_{23} = g(y) \quad \text{and} \quad p_{33} = h(y) \quad \text{at} \quad z = 0 \tag{5.16}$$

We write

$$g(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{g}(k)\, e^{-iky}\, dk, \quad \bar{g}(k) = \int_{-\infty}^{\infty} g(y)\, e^{iky}\, dy \tag{5.17}$$

$$h(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{h}(k)\, e^{-iky}\, dk, \quad \bar{h}(k) = \int_{-\infty}^{\infty} h(y)\, e^{iky}\, dy \tag{5.18}$$

Equations (4.23), (4.24), (5.14) - (5.18) yield

$$A_p = \frac{1}{\Omega k^2 |k|} \left[\bar{g}(M_{43} + M_{44})\, ki - \bar{h}(M_{33} + M_{34})\, |k| \right] \tag{5.19}$$

$$C_p = \frac{1}{\Omega k^2 |k|} \left[\bar{h}(M_{31} + M_{32})|k| - \bar{g}(M_{41} + M_{42})\, ki \right] \tag{5.20}$$

where

$$\Omega = (M_{31} + M_{32})(M_{43} + M_{44}) - (M_{33} + M_{34})(M_{41} + M_{42}) \tag{5.21}$$

The integral expressions for the displacements at any point of the mth layer can be found from (4.21), (4.22), (5.6), (5.19) and (5.20). We obtain

$$v = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [(N_{11} + N_{12})(\{\bar{g}(M_{43} + M_{44})\, ki - \bar{h}(M_{33} + M_{34})|k|\}) + (N_{13}+N_{14})(\{\bar{h}(M_{31}+M_{32})|k| - \bar{g}(M_{41}+M_{42})\, ki\})]\, e^{-iky}\, \Omega^{-1} (k|k|)^{-1}\, dk \quad \ldots.(5.22)$$

$$w = \frac{1}{2\pi} \int_{-\infty}^{\infty} [(N_{21}+N_{22})(\{\bar{g}(M_{43}+M_{44})\, ki - \bar{h}(M_{33}+M_{34})|k|\}) + (N_{23}+N_{24})(\{\bar{h}(M_{31}+M_{32})|k| - \bar{g}(M_{41}+M_{42})\, ki\})]\, e^{-iky}\, \Omega^{-1} k^{-2}\, dk \quad \ldots.(5.23)$$

The stresses for the plane strain problem can be obtained from (4.2) - (4.5), (5.22) and (5.23).

6. STRIP LOADING AT THE BOUNDARY $z = 0$

The results obtained in the previous section are of general nature. We now consider a few particular cases in which the surface load is precisely defined.

6.1 Uniform Normal Pressure

Suppose that a uniform normal pressure p acts over a segment $-a \leq y \leq a$ of the y-axis in the positive direction of the z-axis (Fig. 2). This is a plain strain problem and the boundary conditions at $z = 0$ are

$$p_{23} = 0, \qquad p_{23} = \begin{cases} -p & |y| \leq a \\ 0 & |y| > a \end{cases} \tag{6.1}$$

From (5.16) - (5.18) and (6.1), we find

$$\overline{g}(k) = 0, \qquad \overline{h}(k) = -2p \left(\frac{\sin ka}{k}\right) \tag{6.2}$$

Equation (5.22), (5.23) and (6.2) give the integral expressions for displacements at any point of the mth layer. We obtain

$$v = \frac{p}{\pi i} \int_{-\infty}^{\infty} \sin ka \; [(N_{11}+N_{12})(M_{33}+M_{34}) - (N_{13}+N_{14})(M_{31}+M_{32})] \, e^{-iky} \, \Omega^{-1} \, k^{-2} \, dk \qquad \ldots(6.3)$$

$$w = \frac{p}{\pi} \int_{-\infty}^{\infty} \sin ka \; [(N_{21}+N_{22})(M_{33}+M_{34}) - (N_{23}+N_{24})(M_{31}+M_{32})] \, e^{-iky} \, \Omega^{-1} \, k^{-3} \, |k| \, dk \qquad \ldots(6.4)$$

6.2 Tangential Load

Let a uniform load Q per unit area act over the strip $|y| \leq a$ of the boundary in the positive direction of the y-axis. This problem is again a plain strain problem and the boundary conditions are

$$\bar{p}_{23} = \begin{cases} -Q & |y| < a, \\ 0 & |y| > a, \end{cases} \qquad p_{33} = 0 \tag{6.5}$$

From (5.16) - (5.18) and (6.5), we obtain

$$\overline{g}(k) = -2Q \left(\frac{\sin ka}{k}\right), \qquad \overline{h}(k) = 0 \tag{6.6}$$

The integral expressions for displacements can be obtained from (5.22), (5.23) and (6.6), we get

$$v = \frac{Q}{\pi} \int_{-\infty}^{\infty} \sin ka \; [(N_{13}+N_{14})(M_{41}+M_{42}) - (N_{11}+N_{12})(M_{43}+M_{44})] e^{-iky}$$

$$\Omega^{-1} \; (k|k|)^{-1} \; dk \qquad (6.7)$$

$$w = \frac{-Q}{\pi i} \int_{-\infty}^{\infty} \sin ka \; [(N_{23}+N_{24})(M_{41}+M_{42}) - (N_{21}+N_{22})(M_{43}+M_{44})] e^{-iky}$$

$$\Omega^{-1} \; k^{-2} \, dk \qquad ...(6.8)$$

6.3 Shear Load

Assume that a uniform load R per unit area is acting over the strip $|y| \leq a$ of the surface $z = 0$ in the positive x-direction (Fig. 3). This is an antiplane strain problem and boundary condition at $z = 0$ is

$$p_{13} = \begin{cases} -R & |y| \leq a \\ 0 & |y| > a \end{cases} \qquad (6.9)$$

From (5.9), (5.11) and (6.9), we obtain

$$\bar{f}(k) = -2R \left(\frac{\sin ka}{k}\right) \qquad (6.10)$$

Equations (5.13) and (6.10) yield

$$u = \frac{-R}{\pi} \int_{-\infty}^{\infty} \sin ka \left(\frac{N_{11}+N_{12}}{M_{21}+M_{22}}\right) e^{-iky} \, (k|k|)^{-1} \, dk \qquad (6.11)$$

In this section, we have obtainedintegral expressions for the displacements for three types of surface loading. The stresses can be obtained by using the usual stress-diaplacement relations.

7. DEFORMATION OF A UNIFORM HALF-SPACE BY STRIP LOADING

For a half-space, $p = 1$ and

$$[M] = [Z(0)] \quad \text{and} \quad N = [Z(z)] \qquad (7.1)$$

7.1 Uniform Normal Pressure

In this case, from (4.27) and (7.1);

$$M = [Z(0)] = \begin{bmatrix} \frac{-1}{2\pi} & 0 & 0 & \frac{1}{2\pi\alpha} \\ 0 & -\frac{1}{2\mu} & \frac{1}{2\mu}\left(\frac{1}{\alpha}-1\right) & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (7.2)$$

$N = [Z(z)]$ is given in (4.27) and

$$\Omega = -1 \qquad (7.3)$$

The integral expressions for the displacements at any point of the uniform half-space due to normal pressure P at the boundary z = 0 are obtained from (4.27), (6.3), (6.4), (7.2) and (7.3). We find

$$v = \frac{P}{\pi\mu} \int_0^\infty \left(1 - \frac{1}{\alpha} + kz\right) e^{-kz} \sin ka \sin ky\, k^{-2}\, dk \qquad (7.4)$$

$$w = \frac{P}{\pi\mu} \int_0^\infty \left(\frac{1}{\alpha} + kz\right) e^{-kz} \sin ka \cos ky\; k^{-2}\, dk \qquad (7.5)$$

Using (4.2) - (4.5), the stresses are found to be

$$p_{11} = \frac{2P(1-2\alpha)}{\alpha\pi} \int_0^\infty e^{-kz} \sin ka \cos ky\, k^{-1}\, dk$$

$$= \frac{P(1-2\alpha)}{\pi\alpha} \tan^{-1}\left(\frac{2az}{y^2+z^2-a^2}\right) \qquad \ldots\ldots(7.6)$$

$$p_{22} = \frac{-2P}{\pi} \int_0^\infty \left(\frac{1-kz}{k}\right) e^{-kz} \sin ka \cos ky\, dk$$

$$= \frac{-2P}{\pi} \left[\frac{1}{2} \tan^{-1}\left(\frac{2az}{y^2+z^2-a^2}\right) - \frac{az\,(z^2+a^2-y^2)}{[z^2+(a+y)^2][z^2+(a-y)^2]}\right]$$

$$\ldots\ldots(7.7)$$

$$p_{33} = \frac{-2P}{\pi} \int_0^\infty \left(\frac{1+kz}{k}\right) e^{-kz} \sin ka \cos ky \, dk$$

$$= \frac{-2P}{\pi} \left[\frac{1}{2} \tan^{-1}\left(\frac{2az}{y^2+z^2-a^2}\right) + \frac{az\,(z^2+a^2-y^2)}{[z^2+(a+y)^2]\,[z^2+(a-y)^2]}\right] \qquad \ldots\ldots(7.8)$$

$$p_{23} = \frac{-2Pz}{\pi} \int_0^\infty e^{-kz} \sin ka \sin ky \, dk$$

$$= \frac{-4Pa}{\pi} \left[\frac{y z^2}{[z^2 + (a+y)^2]\,[z^2 + (a-y)^2]}\right] \qquad \ldots(7.9)$$

The integrals expressing the stresses have been evaluated with the help of standard integrals listed in Appendix-II. The closed form expressions for the stresses given in (7.7) - (7.9) coincide with the corresponding expressions given by Sneddon (1951; p. 407).

The problem of the normal line load, say P_0, per unit length applied at the boundary of the uniform half-space becomes the particular case of normal strip-loading problem. Taking $[P = (P_0/2a)]$ and using the relation

$$\lim_{a \to 0} \left(\frac{\sin ka}{ka}\right) = 1 \qquad (7.9a)$$

in (7.4) and (7.5), we obtain the closed form expressions for the displacements as given below :

$$v = \frac{P_0}{2\pi\mu} \int_0^\infty \left(1 - \frac{1}{\alpha} + kz\right) e^{-kz} \sin ky \; k^{-1} \, dk$$

$$= \frac{P_0}{2\pi\mu} \left[\left(\frac{\alpha-1}{\alpha}\right) \tan^{-1}\left(\frac{y}{z}\right) + \frac{yz}{y^2+z^2}\right] \qquad (7.10)$$

$$w = \frac{P_0}{2\pi\mu} \int_0^\infty \left(\frac{1}{\alpha} + kz\right) e^{-kz} \cos ky \; k^{-1} \, dk$$

$$= \frac{P_0}{2\pi\mu} \left[\frac{-1}{2\alpha} \log(y^2+z^2) + \frac{z^2}{y^2+z^2}\right] \qquad (7.11)$$

The stresses follow directly from (7.6) - (7.9) :

$$p_{11} = \frac{P_0\,(1-2\alpha)}{\pi\alpha}\left[\frac{z}{y^2+z^2}\right], \tag{7.12}$$

$$p_{22} = \frac{-2\,P_0}{\pi}\left[\frac{zy^2}{(y^2+z^2)^2}\right], \tag{7.13}$$

$$p_{33} = \frac{-2\,P_0}{\pi}\left[\frac{z^3}{(y^2+z^2)^2}\right], \tag{7.14}$$

$$p_{23} = \frac{-2\,P_0}{\pi}\left[\frac{yz^2}{(y^2+z^2)^2}\right] \tag{7.15}$$

The stresses (7.13) - (7.15) coincide with the corresponding results of Sneddon (1951; p. 409). We observe that Sneddon (1951) did not obtain the closed form expression for displacements.

7.2 Tangential Load

From (6.7), (6.8), (7.2) and (7.3), we get

$$v = \frac{Q}{\pi\mu\alpha}\int_0^\infty (1-\alpha kz)\,e^{-kz}\,\sin ka\,\cos ky\,k^{-2}\,dk, \tag{7.16}$$

$$w = \frac{Q}{\pi\mu\alpha}\int_0^\infty (1-\alpha+\alpha kz)\,e^{-kz}\,\sin ka\,\sin ky\,k^{-2}\,dk \tag{7.17}$$

Using (4.2) - (4.5), (7.16) and (7.17), the stresses are found to be

$$p_{11} = \frac{2Q}{\pi}\left(\frac{1-2\alpha}{\alpha}\right)\int_0^\infty k^{-1}\,e^{-kz}\,\sin ka\,\sin ky\,dk$$

$$= \frac{Q}{2\pi}\left(\frac{1-2\alpha}{\alpha}\right)\left[\log\left\{\frac{(a+y)^2+z^2}{(a-y)^2+z^2}\right\}\right. \qquad .....(7.18)$$

$$p_{22} = \frac{-2Q}{\pi} \int_0^{\infty} \left(\frac{2-kz}{k}\right) e^{-kz} \sin ka \sin ky \, dk$$

$$= \frac{-Q}{\pi} \left[\log \frac{(a+y)^2 + z^2}{(a-y)^2 + z^2} - \frac{4\,ayz^2}{[(a+y)^2+z^2][(a-y)^2+z^2]}\right] \quad (7.19)$$

$$p_{33} = \frac{-2Qz}{\pi} \int_0^{\infty} e^{-kz} \sin ka \sin ky \, dk$$

$$= \frac{-4Q}{\pi} \left[\frac{ayz^2}{[(a+y)^2+z^2][(a-y)^2+z^2]}\right] \quad (7.20)$$

$$p_{23} = \frac{-2Q}{\pi} \int_0^{\infty} \left(\frac{1-kz}{k}\right) e^{-kz} \sin ka \cos ky \, dk$$

$$= \frac{-2Q}{\pi} \left[\frac{1}{2} \tan^{-1}\left(\frac{2az}{y^2+z^2-a^2}\right) - \frac{a(z^2+a^2-y^2)\,z}{[(a+y)^2+z^2][(a-y)^2+z^2]}\right]$$

.....(7.21)

Taking $[Q = (Q_0/2a)]$ and proceeding to the limit $a \longrightarrow 0$ in (7.16) and (7.17), we obtain the deformation field due to a tangential line load Q_0, per unit length acting at the boundary $z = 0$ in the positive y-direction.

$$v = \frac{Q_0}{2\pi\mu\alpha} \int_0^{\infty} (1-\alpha kz)\, e^{-kz} \cos ky \, k^{-1} \, dk$$

$$= \frac{Q_0}{2\pi\mu\alpha} \left[\frac{-1}{2} \log(y^2+z^2) - \frac{\alpha z^2}{y^2+z^2}\right] \quad (7.22)$$

$$w = \frac{Q_0}{2\pi\mu\alpha} \int_0^{\infty} (1-\alpha+\alpha kz)\, e^{-kz} \sin ky \, k^{-1} \, dk$$

$$= \frac{Q_0}{2\pi\mu\alpha} \left[(1-\alpha)\tan^{-1}(y/z) + \frac{\alpha yz}{y^2+z^2}\right] \quad (7.23)$$

The stresses follow directly from (7.18) - (7.21). We get

$$p_{11} = \frac{Q_0}{\pi} \left(\frac{1-2\alpha}{\alpha}\right) \frac{y}{(y^2+z^2)} \tag{7.24}$$

$$p_{22} = \frac{-2\,Q_0}{\pi} \frac{y^3}{(y^2+z^2)^2} \tag{7.25}$$

$$p_{33} = \frac{-2\,Q_0}{\pi} \frac{y\,z^2}{(y^2+z^2)^2} \tag{7.26}$$

$$p_{23} = \frac{-2\,Q_0}{\pi} \frac{y^2\,z}{(y^2+z^2)^2} \tag{7.27}$$

The stresses p_{23} and p_{33} coincide with the corresponding results obtained by Garg and Singh (1985). They had obtained the stresses p_{23} and p_{33} only while we have found the closed form expressions for all the displacements and stresses.

7.3 Anti Plane Strain Problem

In this case

$$M = [Z(0)] = \begin{bmatrix} 1 & 0 \\ 0 & -\mu \end{bmatrix} \tag{7.28}$$

and $N = [Z(z)]$ is given in (3.14).

From (3.14), (6.11) and (7.28), we obtain

$$u = \frac{2R}{\pi\mu} \int_0^\infty e^{-kz} \sin ka \cos ky\, k^{-2}\, dk \tag{7.29}$$

Equations (3.2) and (7.29) give the stresses for the antiplane strain problem. We get

$$p_{12} = \frac{-2R}{\pi} \int_0^\infty e^{-kz} \sin ka \; \sin ky \; k^{-1} \, dk$$

$$= \frac{-R}{2} \log \frac{(a+y)^2 + z^2}{(a-y)^2 + z^2} \qquad (7.30)$$

$$p_{13} = \frac{-2R}{\pi} \int_0^\infty e^{-kz} \sin ka \; \cos ky \; k^{-1} \, dk$$

$$= \frac{-R}{\pi} \tan^{-1} \left[\frac{2az}{y^2+z^2-a^2} \right] \qquad (7.31)$$

Taking $R = (R_0/2a)$ and proceeding to the limit $a \longrightarrow 0$ in (7.29)-(7.31), we obtain the deformation field caused by a shear line load R_0, per unit length, acting at the boundary $z = 0$ in the positive x-direction.

$$u = \frac{R_0}{\pi\mu} \int_0^\infty e^{-kz} \cos ky \; k^{-1} \, dk$$

$$= \frac{-R_0}{2\pi\mu} \log (y^2 + z^2) \qquad (7.32)$$

$$p_{12} = \frac{-R_0}{\pi} \left[\frac{y}{y^2 + z^2} \right] \qquad (7.33)$$

and

$$p_{13} = \frac{-R_0}{\pi} \left[\frac{z}{y^2 + z^2} \right] \qquad (7.34)$$

The expression for the stress p_{13} is the same as obtained by Garg and Singh (1985). Here, we have obtained the entire deformation field corresponding to the antiplane strain problem.

8. A LAYER OF THICKNESS d LYING OVER A HALF-SPACE

In this section, we consider the antiplane strain problem only (Fig. 4). In this case

$$M = [a_1 (d)] \; [Z_2 (d)] \tag{8.1}$$

$$N_1 = [a_1 (d-z)] \; [Z_2 (d)] \quad \text{for} \quad 0 \leqslant z \leqslant d \tag{8.2}$$

$$N_2 = [Z_2 (z)] \quad \text{for} \quad z \leqslant d \tag{8.3}$$

The displacement u at any point of the layer is obtained from (6.11), (8.1) and (8.2). Expressing the hyperbolic functions in terms of exponential functions and expanding the denominator in a power series, we find

$$u = \frac{R}{\pi \mu_1} \sum_{-\infty}^{\infty} \frac{\sin ka}{k|k|} \left[e^{-|k|z} + \sum_{n=1}^{\infty} C^n \{e^{-|k|(2nd+z)} + e^{-|k|(2nd-z)}\} \right. e^{-iky} \, dk$$

$$= \frac{2R}{\pi \mu_1} \int_0^{\infty} \frac{\sin ka \cos ky}{k^2} \left[e^{-kz} + \sum_{n=1}^{\infty} C^n \{e^{-k(2nd+z)} + e^{-k(2nd-z)}\}\right] dk$$

....(8.4)

where

$$C = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \tag{8.5}$$

The non-zero stresses p_{12} and p_{13} are found to be

$$p_{12} = -\frac{2R}{\pi} \int_0^{\infty} \frac{\sin ka \sin ky}{k} \left[e^{-kz} + \sum_{n=1}^{\infty} C^n (e^{-k(2nd+z)} + e^{-k(2nd-z)})\right] dk$$

$$= -\frac{R}{2\pi} \left[\log \frac{(a+y)^2 + z^2}{(a-y)^2 + z^2} + \sum_{n=1}^{\infty} \left\{ \log \left(\frac{(a+y)^2 + (2nd-z)^2}{(a-y)^2 + (2nd-z)^2} \right) \right.\right.$$

$$\left.\left. + \log \left(\frac{(a+y)^2 + (2nd+z)^2}{(a-y)^2 + (2nd+z)^2} \right) \right\}\right] \quad ....(8.6)$$

$$p_{13} = \frac{2R}{\pi} \int_0^\infty \frac{\sin ka \cos ky}{k} \left[-e^{-kz} + \sum_{n=1}^{\infty} C^n \{e^{-k(2nd-z)} - e^{-k(2nd+z)}\} \right] dk$$

$$= \frac{-R}{\pi} \left[\tan^{-1} \left(\frac{2az}{y^2+z^2-a^2}\right) - \sum_{n=1}^{\infty} C^n \left\{ \tan^{-1} \left(\frac{2a\,(2nd-z)}{y^2-a^2+(2nd-z)^2}\right) \right.\right.$$

$$\left.\left. - \tan^{-1} \left(\frac{2a\,(2nd+z)}{y^2-a^2+(2nd+z)^2}\right) \right\} \right] \quad ....(8.7)$$

in which the integrals have been evaluated by using the Appendix II. Taking $R = (R_0/2a)$ and then using (7.9a), the displacement and the stresses caused by a shear line load R_0, per unit length, acting at the boundary $z = 0$ of the semi-infinite elastic medium in the positive x-direction are found to be

$$u = \frac{R_0}{\pi\mu_1} \int_0^\infty k^{-1} \cos ky \left[e^{-kz} + \sum_{n=1}^{\infty} C^n \{(e^{-k(2nd+z)} + e^{-k(2nd-z)})\} \right] dk$$

$$= \frac{-R_0}{2\pi\mu_1} \left[\log (y^2+z^2) + \sum_{n=1}^{\infty} C^n \{\log (y^2+(2nd+z)^2) + \log (y^2+(2nd-z)^2)\} \right] \quad ....(8.8)$$

$$p_{12} = \frac{-R_0}{\pi} \left[\frac{y}{y^2+z^2} + \sum_{n=1}^{\infty} C^n \left\{ \frac{y}{y^2+(2nd+z)^2} + \frac{y}{y^2+(2nd-z)^2} \right\} \right] \quad ....(8.9)$$

$$p_{13} = \frac{-R_0}{\pi} \left[\frac{z}{y^2+z^2} + \sum_{n=1}^{\infty} C^n \left\{ \frac{2nd+z}{y^2+(2nd+z)^2} - \frac{2nd-z}{y^2+(2nd-z)^2} \right\} \right] \quad ....(8.10)$$

The stresses in (8.9) and (8.10) coincide with the corresponding results obtained by Garg and Singh (1989). Here, these stresses have been obtained by using transfer matrix approach while Garg and Singh (1989) obtained them directly. The advantage of our method is that it can also be used when there is more than one layer lying over the half-space.

9. DISCUSSION

In Section 6, we have obtained the deformation field at any point of the multilayered medium caused by normal and shear strip-loading. These results are in the form of integrals over the variable k. These integrals can be evaluated by using the method suggested by Jovanovich et al. (1974). The closed form expressions for a uniform half-space given in Section 7 and for a layer over a half-space given in Section 8 can be used as a check over the numerical computation. The quasi-static deformation of a uniform half-space and a single layer lying over a half-space can be obtained by the simple and straightward procedure given by Garg and Singh (1989) and Singh and Singh (1989).

10. REFERENCES

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APPENDIX - I

(1) Plane Problem

$$[Z(0)]^{-1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 2\mu\alpha & (1-\alpha) & 0 \\ 0 & 2\mu\alpha & -\alpha & 0 \\ 2\mu\alpha & 0 & 0 & -\alpha \end{bmatrix}$$

Elements of the transfer matrix $[a_m]$ are (omitting the subscript m);

$(11) = (33) = \mathrm{ch}|k|d + |k|d\ \mathrm{sh}\ |k|\ d$

$(12) = -(43) = \alpha|k|d\ \mathrm{ch}|k|d + (1-\alpha)\mathrm{sh}\ |k|d$

$(13) = -\frac{1}{2\mu} = [\alpha|k|d\ \mathrm{ch}|k|d + (2-\alpha)\mathrm{sh}\ |k|d]$

$(14) = -(23) = [-\alpha|k|d / 2\mu]\ \mathrm{sh}|k|d$

$(21) = -(34) = -\alpha|k|d\ \mathrm{ch}|k|d + (1-\alpha)\mathrm{sh}|k|d$

$(22) = (44) = \mathrm{ch}|k|d - \alpha|k|d\ \mathrm{sh}|k|d$

$(24) = \frac{1}{2\mu}[\alpha|k|d\ \mathrm{ch}|k|d + (\alpha-2)\ \mathrm{sh}|k|d\ ]$

$(31) = -2\mu\alpha\,[|k|d\ \mathrm{ch}|k|d + \mathrm{sh}|k|d]$

$(32) = -(41) = -2\mu\alpha|k|d\ \mathrm{sh}|k|d$

$(42) = 2\mu\alpha\ [|k|d\ \mathrm{ch}|k|d - \mathrm{sh}|k|d]$

(2) Antiplane Problem

$$[a_m] = \begin{bmatrix} \mathrm{ch}\,(|k|d_m) & -\mu_m^{-1}\ \mathrm{sh}\,(|k|d_m) \\ -\mu_m\,\mathrm{sh}\,(|k|d_m) & \mathrm{ch}\,(|k|d_m) \end{bmatrix}$$

APPENDIX - II

1. $\int_0^\infty k^{-1} e^{-kz} \cos ky \, dk = -\frac{1}{2} \log (y^2 + z^2)$

2. $\int_0^\infty k^{-1} e^{-kz} \sin ky \, dk = \tan^{-1} (y/z)$

3. $\int_0^\infty e^{-kz} \cos ky \, dk = z / (y^2 + z^2)$

4. $\int_0^\infty e^{-kz} \sin ky \, dk = y / (y^2 + z^2)$

5. $\int_0^\infty k e^{-kz} \sin ky \, dk = 2yz / (y^2 + z^2)^2$

6. $\int_0^\infty k e^{-kz} \cos ky \, dk = (z^2 - y^2) / (y^2 + z^2)^2$

7. $\int_0^\infty e^{-kz} \sin ka \sin ky \, dk = \dfrac{2ayz}{[z^2 + (a+y)^2]\,[z^2 + (a-y)^2]}$

8. $\int_0^\infty e^{-kz} \sin ka \cos ky \, dk = \dfrac{a\,(z^2 + a^2 - y^2)}{[z^2 + (a+y)^2]\,[z^2 + (a-y)^2]}$

9. $\int_0^\infty k^{-1} e^{-kz} \sin ka \sin ky \, dk = \frac{1}{4} \log \left[\dfrac{(a+y)^2 + z^2}{(a-y)^2 + z^2}\right]$

10. $\int_0^\infty k^{-1} e^{-kz} \sin ka \cos ky \, dk = \frac{1}{2} \tan^{-1} [(2az) / (y^2 + z^2 - a^2)]$

In relation 10, it is assumed that $y^2+z^2 \geq a^2$. However, if $y^2+z^2 < a^2$, we must add $(\pi/2)$ to the right side [Gradshteyn and Ryzhik, 1980, p. 492].

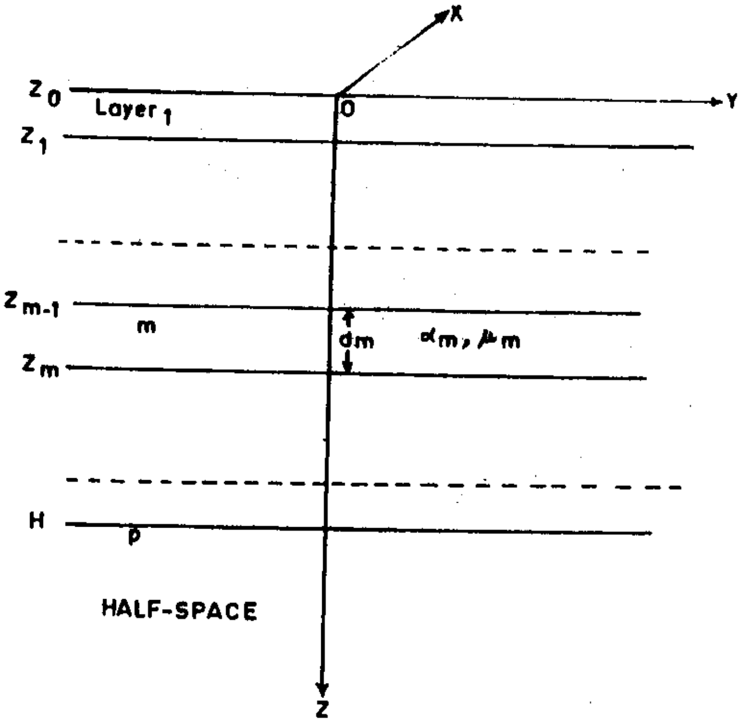

Fig. 1. Multilayered half-space

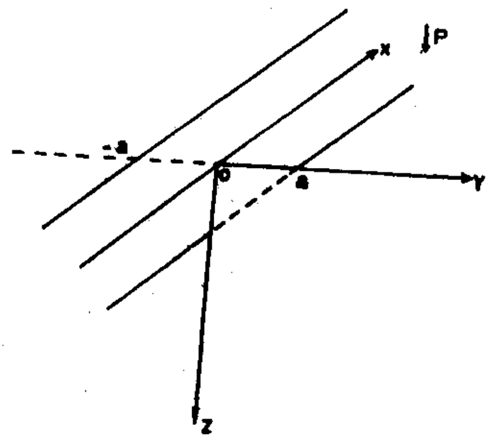

Fig.2. Normal strip-loading on the boundary of a semi-infinite medium

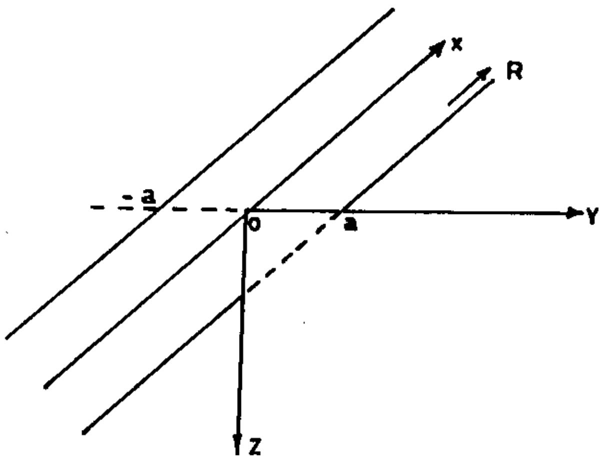

Fig. 3. Shear strip-loading on the boundary of a semi-infinite medium

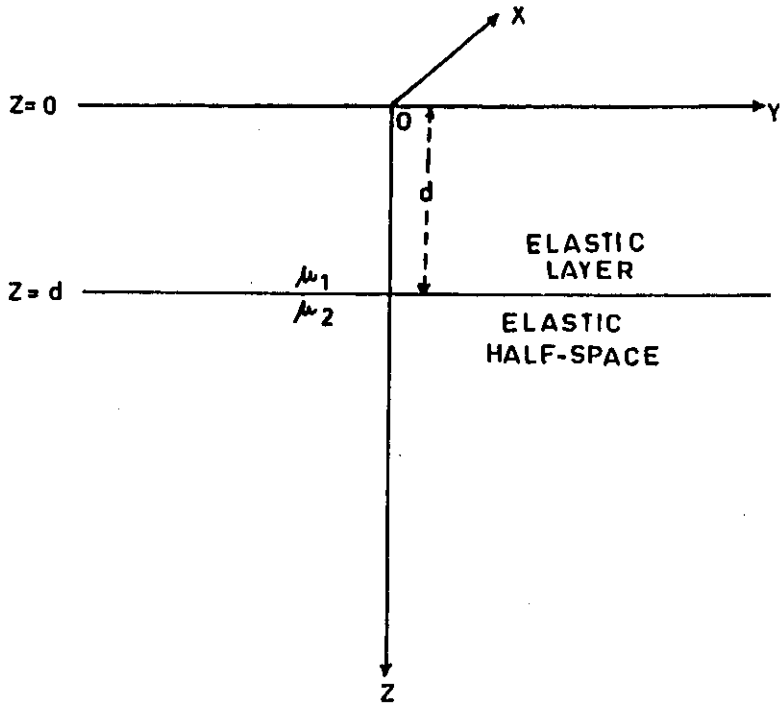

Fig. 4. A layer of thickness d lying over a half-space