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RAYLEIGH WAVES IN A THREE LAYERED VISCOELASTIC MEDIUM[1]

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Introduction

The anelastic nature of the earth materials and the fact that the theory of viscoelastic describes the linear behaviour of both elastic and anelastic materials, suggested to Buchen [1971] and Borcherdt [1973 a, b] to study the propagation of waves in homogeneous, isotropic, linear viscoelastic materials. They found a distinct difference between the inhomogeneous waves of elastic media and the inhomogeneous waves of linear anelastic media. Borcherdt [1973] investigated this difference for Rayleigh-type surface waves on a linear viscoelastic half-space and found that the physical properties of the surface waves are distinct from those predicted by perfect elasticity theory. Using general plane wave solutions, we derive the frequency equation for propagation of Rayleigh-type surface waves in a three layered homogeneous, isotropic, viscoelastic medium.

Formulation of the problem

Consider a homogeneous linear viscoelastic medium composed of two layers I and II with finite thicknesses H_1 and H_2 respectively, overlying a half-space. The geometry of the problem is depicted in Fig. 1. Let (x, y, z) be the Cartesian coordinate system such that the two layers I and II are bounded by the surfaces $z = 0$, $z = -H_1$ and $z = 0$, $z = H_2$ respectively, and the third medium is given by $z \geq H_2$.

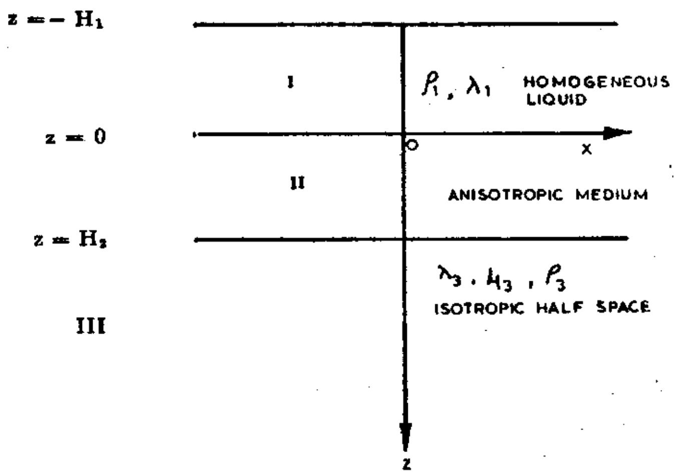

Fig. 1

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Basic Equations

In a homogeneous isotropic linear viscoelastic material the harmonic motions are governed by the equation of motion

$$(\lambda^* + \mu^*) \nabla (\nabla \vec{u}) + \mu^* \nabla^2 \vec{u} + \rho w^2 \vec{u} = 0 \quad (1)$$

where $\vec{u} = (u_x, u_y, u_z)$ is the displacement vector, λ^* and μ^* are frequency dependent complex valued material constants that reduce to the Lame's constants in the elastic case. ρ is the mass density.

Using Helmholtz's theorem, we write

$$\vec{u} = \nabla \phi + \nabla \times \vec{\psi} \quad (2)$$

where ϕ and $\vec{\psi}$ are scalar and vecter potentials respectively such that

$$\nabla \vec{\psi} = 0 \quad (3)$$

The equation of motion (1) is satisfied if

$$\nabla^2 \phi + k_p^2 \phi = 0 \quad (4)$$

$$\nabla^2 \vec{\phi} + k_s^2 \vec{\psi} = 0 \quad (5)$$

where k_p^2 and k_s^2 are complex quantities defined by

$$k_p^2 = w^2/\alpha^{*2} \quad (6)$$

$$k_s^2 = w^2/\beta^{*2} \quad (7)$$

and

$$\alpha^* = \text{p.v} \left[\frac{\lambda^* + 2\mu^*}{\rho} \right]^{\frac{1}{2}} \quad (8)$$

$$\beta^* = \text{p.v.} [\mu^*/\rho]^{1/2} \quad (9)$$

are complex velocities, where p.v denotes the principal value.

The plane wave solutions of (4) and (5) for the displacement potentials to describe the two dimensional surface waves are,

$$\phi_l = \sum_{j=1}^{2} B_{lj} \exp [-\vec{A}\phi_{lj}. \vec{r}] \exp [i(wt - \vec{P}\phi_{lj}. \vec{r})] \quad (10)$$

$$\psi_l = \sum_{j=1}^{2} C_{lj} \hat{n}_{lj} \exp [-\vec{A}\psi_{lj}. \vec{r}] \exp [i(wt - \vec{P}\psi_{lj}. \vec{r})] \quad (11)$$

where $l = 1, 2$ correspond to the media I and II respectively, and $j = 1, 2$ to downward and upward going waves, $\hat{n}$ is taken to be a unit vector in the y-direction, $\vec{r}$ is the position vector and

$$\vec{P}\phi_{lj} = k_R \hat{x} + (-1)^{j+1} b\alpha_{l*I} \hat{z} \tag{12}$$

$$\vec{P}\psi_{lj} = k_R \hat{x} + (-1)^{j+1} b\beta_l{}^*{}_I \hat{z} \tag{13}$$

$$\vec{A}\phi_{lj} = -k_I \hat{x} + (-1)^{j+1} b\alpha_{l*R} \hat{z} \tag{15}$$

$$\vec{A}\psi_{lj} = -k_I \hat{x} + (-1)^{j+1} b\beta_{l*R} \hat{z} \tag{16}$$

are the propagation and attenuation vectors which satisfy the following conditions

$$\vec{P}\phi_{lj} . \vec{P}\phi_{lj} - \vec{A}\phi_{lj} . \vec{A}\phi_{lj} = \mathrm{Re}\,[k_{pl}{}^2] \tag{16}$$

$$\vec{P}\phi_{lj} . \vec{A}\phi_{lj} = -\tfrac{1}{2}\,\mathrm{Im}\,[k_{pl}{}^2] \tag{17}$$

$$\vec{P}\psi_{lj} . \vec{P}\psi_{lj} - \vec{A}\psi_{lj} . \vec{A}\psi_{lj} = \mathrm{Re}\,[k_{sl}{}^2] \tag{18}$$

$$\vec{P}\psi_{lj} . \vec{A}\psi_{lj} = -\tfrac{1}{2}\,\mathrm{Im}\,[k_{sl}{}^2] \tag{19}$$

The complex wave number $k = k_R + i\,k_I$ is chosen such that $k_R > 0$, where the subscript R denotes the real part and the subscript I the imaginary part, and

$$b\alpha_{l*} = \text{p.v.}\ [k^2 - k_{pl}{}^2]^{1/2} \tag{20}$$

$$b\beta_{l*} = \text{p.v.}\ [k^2 - k_{sl}{}^2]^{1/2} \tag{21}$$

In the third medium i.e., in the region $z > H_2$, we have

$$\phi = B_{31} \exp[-\vec{A}\phi . \vec{r}] \exp[i(wt - \vec{P}\phi . \vec{r})] \tag{22}$$

$$\vec{\psi} = C_{31}\hat{n} \exp[-\vec{A}\psi . \vec{r}] \exp[i(wt - \vec{P}\psi . \vec{r})] \tag{23}$$

where the propagation and attenuation vectors are defined by

$$\vec{P}\phi = k_R \hat{x} + b_{\alpha * I} \hat{z} \tag{24}$$

$$\vec{P}\psi = k_R \hat{x} + b_{\beta * I} \hat{z} \tag{25}$$

$$\vec{A}\phi = -k_I \hat{x} + b_{\alpha * R} \hat{z} \tag{26}$$

$$\overrightarrow{A\psi} = -k_I\,\hat{x} + b_{\beta^*_R}\,\hat{z} \tag{27}$$

and satisfy the following condition,

$$\overrightarrow{P\phi}\cdot\overrightarrow{P\phi} - \overrightarrow{A\phi}\cdot\overrightarrow{A\phi} = \mathrm{Re}\,[k_p^2] \tag{28}$$

$$\overrightarrow{P\phi}\cdot\overrightarrow{A\phi} = -\tfrac{1}{2}\,\mathrm{Im}\,[k_p^2] \tag{29}$$

$$\overrightarrow{P\psi}\cdot\overrightarrow{P\psi} - \overrightarrow{A\psi}\cdot\overrightarrow{A\psi} = \mathrm{Re}\,[k_s^2] \tag{30}$$

$$\overrightarrow{P\psi}\cdot\overrightarrow{A\psi} = -\tfrac{1}{2}\,\mathrm{Im}\,[k_s^2] \tag{31}$$

Boundary Conditions and the Period Equations

The boundary conditions, that the displacements and stresses are continuous at the interfaces $z = 0$ and $z = H_2$, and that the stresses vanish at the free surface $z = -H_1$, yield the following system of ten homogeneous equations represented by the matrix equation

$$Q\,X = 0 \tag{32}$$

where Q is the 10×10 matrix

$$Q = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} \tag{33}$$

with

$$D_1 = \begin{bmatrix} (2k^2-k^2_{s1})\,e^{b\alpha_1^* H_1} & 2kb\beta_1^*\,e^{b\beta_1^* H_1} & (2k^2-k^2_{s1})\,e^{-b\alpha_1^* H_1} & -2kb\beta_1^*\,e^{-b\beta_1^* H_1} \\ 2k\,b\alpha_1^*\,e^{b\alpha_1^* H_1} & (2k^2-k^2_{s1})\,e^{b\beta_1^* H_1} & -2k\,b\alpha_1^*\,e^{-b\alpha_1^* H_1} & (2k^2-k^2_{s1})\,e^{-b\beta_1^* H_1} \\ -k & -b\beta_1^* & -k & b\beta_1^* \\ -b\alpha_1^* & -k & b\alpha_1^* & -k \end{bmatrix} \tag{34}$$

$$D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k & b\beta_2^* & k & -b\beta_2^* & 0 & 0 \\ b\alpha_2^* & k & b\alpha_2^* & k & 0 & 0 \end{bmatrix} \tag{35}$$

$$D_3 = \begin{bmatrix} (2k^2-k^2_{s1}) & 2kb\beta_1^* & (2k^2-k_{s1}^2) & -2kb\beta_1^* \\ 2k\,b\alpha_1^* & (2k^2-k^2_{s1}) & -2k\,b\alpha_1 & (2k^2-k_{s1}^2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

$$D_4 = \begin{vmatrix} -\frac{\mu_2^*}{\mu_1^*}(2k^2-k_{s2}^2) & -2k\frac{\mu_2^*}{\mu_1^*} & -\frac{\mu_2^*}{\mu_1^*}(2k^2-k_{s2}^2) & 2k\frac{\mu_2^*}{\mu_1^*}b\beta_2^* & 0 & 0 \\ -\frac{2\mu_2^*}{\mu_1^*}k\,b\alpha_2^* & -\frac{\mu_2^*}{\mu_1^*}(2k^2-k_{s2}^2) & \frac{2\mu_2^*}{\mu_1^*}k\,b\alpha_2^* & -\frac{\mu_2^*}{\mu_1^*}(2k^2-k_{s2}^2) & 0 & 0 \\ -k\,e^{-b\alpha_2^*H_2} & -b\beta_2^*\,e^{-\beta_2^*H_2} & -k\,e^{b\alpha_2^*H_2} & b\beta_2^*e^{b\beta^*H_2} & k\,e^{-b\alpha_3^*H_2} & b\beta_3^*\,e^{-b\beta_3^*H_2} \\ -b\alpha_2^*\,e^{-b\alpha_2^*H_2} & -k\,e^{-b\beta_2^*H_2} & b\alpha_2^*\,e^{b\alpha_2^*H_2} & -k\,e^{b\beta_2^*H_2} & b\alpha_3^*\,e^{-b\alpha_3^*H_2} & k\,e^{-b\beta_3^*H_2} \\ 2k\,b\alpha_2^*\,e^{-b\alpha_2^*H_2} & (2k^2-k_{s2}^2)\,e^{-b\beta_2^*H_2} & -2kb\alpha_2^*\,e^{b\alpha_2^*H_2} & (2k^2-k_{s2}^2)\,e^{b\beta_2^*H_2} & -2k\frac{\mu_3^*}{\mu_2^*}b\alpha_3^*\,e^{-b\alpha_3^*H_2} & -\frac{\mu_3^*}{\mu_2^*}(2k^2-k_{s3}^2)\,e^{-b\beta_3^*H_2} \\ (2k^2-k_{s2}^2)\,e^{-b\alpha_2^*H_2} & 2k\,b\beta_2^*\,e^{-b\beta_2^*H_2} & (2k^2-k_{s2}^2)\,e^{b\alpha_2^*H_2} & -2k\,b\beta_2^*\,e^{b\beta_2^*H_2} & -(2k^2-k_{s3}^2)\frac{\mu_3^*}{\mu_2^*}\,e^{-b\alpha_3^*H_2} & -2k\frac{\mu_3^*}{\mu_2^*}b\beta_3^*\,e^{-b\beta_3^*H_2} \end{vmatrix} \quad (37)$$

and X is the column vector

$$X = \{B_{11},\ i\,C_{11},\ B_{12},\ i\,C_{12},\ B_{21},\ i\,C_{21},\ B_{22},\ i\,C_{22},\ B_{31},\ i\,C_{31}\} \quad (38)$$

The condition that the system has a non-zero solution X is that

$$|\,Q\,| = 0 \quad (39)$$

which is the required frequency equation.

Discussion

We shall discuss the frequency equation (39) in the following limiting cases, which afford some verification of the analysis.

(i) The thickness of the layer I is zero, i.e., $H_1 = 0$. Then we obtain the frequency equation of Rayleigh waves in a single layer II overlying a halfspace M_3, the 6-rowed determinantal equation given by

$$|\,D_4\,| = 0, \quad (40)$$

which, in the perfectly elastic case when $\lambda^*(w) = \lambda$ and $\mu^*(w) = \mu$ are Lame's constants, reduces to the frequency equation for Rayleigh waves in a homogeneous perfectly elastic medium with a layer over a halfspace. [See Ewing and Press 1957]. From (40), if we let $H_2 \to \infty$, we find a 4-rowed determinantal equation by deleting first two rows and columns of (40), giving the equation of Stoneley waves at the interface between the media II and III. The frequency equation corresponding to the Rayleigh waves in a layer I can be obtained from (39) by making $H_2 = 0$. In the case when $H_1 \rightarrow \infty$ we obtain an 8-rowed determinantal equation corresponding to the propagation of Rayleigh waves in an internal stratum of thickness H_2, between two semi-infinite media I and III.

(ii) In the case of very short waves, i.e., when $w \rightarrow \infty$, equation (40) factorises into

$$| Q | = | L_1 | \;\; | L_2 | \;\; | L_3 | = 0, \tag{41}$$

where

$$L_1 = \begin{bmatrix} 2k^2 - k^2_{s1} & 2k\, b_{\beta_1*} \\ 2k\, b_{\alpha_1*} & 2k^2 - k^2_{s1} \end{bmatrix} \tag{41}$$

$$L_2 = \begin{bmatrix} -k & b_{\beta_1*} & k & b_{\beta_2*} \\ b_{\beta_1*} & -k & b_{\alpha_2*} & k \\ 2k^2 - k^2_{s1} & -2k\, b_{\beta_1*} & -\frac{\mu_2^*}{\mu_1^*}(2k^2 - k^2_{s1}) & -\frac{2\mu_2^*}{\mu_1^*}\, k\, b_{\beta_1*} \\ 2k\, b_{\alpha_1*} & 2k^2 - k^2_{s1} & -2k\,\frac{\mu^*_2}{\mu^*_1}\, b_{\alpha_2*} & -\frac{\mu_2^*}{\mu_1^*}(2k^2 - k^2_{s2}) \end{bmatrix} \tag{43}$$

$$L_3 = \begin{bmatrix} -k & b_{\beta_2*} & k & b_{\beta_3*} \\ b_{\alpha_2*} & -k & b_{\alpha_3*} & k \\ 2k^2 - k^2_{s2} & -2k\, b_{\beta_2*} & -\frac{\mu_3^*}{\mu^*_2}(2k^2 - k^2_{s3}) & -\frac{\mu_3^*}{\mu_2^*}\, b_{\beta_3*} \\ 2k\, b_{\alpha_2*} & 2k^2 - k^2_{s2} & -2k\,\frac{\mu_3^*}{\mu_2^*}\, b_{\alpha_3*} & -\frac{\mu_3^*}{\mu_2^*}(2k^2 - k^2_{s3}) \end{bmatrix} \tag{44}$$

(41) is equivalent to the equations

$$| L_1 | = 0, \; | L_2 | = 0, \; | L_3 | = 0 \tag{45}$$

which are the frequency equations of Rayleigh and Stoneley type waves for the viscoelastic case for the free surface $z = -H_1$ and the two interfaces $z = 0$ and $z = H_2$. In the limit of perfect elasticity, these three equations reduce to the three equations obtained by Stoneley (1954, p. 614).

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