Paper No. 191, Bulletin ISET, Vol. 16, No. 2, June 1979, pp 43—52

REFLECTION OF SH WAVES FROM A MULTILAYERED ANISOTROPIC CRUST[1]

SOHAN LAL SAINI[2]

Introduction

The problem of the reflection and refraction of elastic waves in the case of an isotropic medium has been discussed extensively (Cf. Ewing et. al. 1957, Brekhovskikh 1960). Transmission of elastic waves at oblique incidence through a stratified solid medium consisting of number of parallel plates of different materials and thickness was studied theoretically by Thomson (1950). He used the matrix calculus to systematize the analysis and to present the equation in a form suitable for computation. Haskell (1953) used matrix method to study dispersion properties of Love and Rayleigh waves in an isotropic homogeneous medium. This technique is now known as Thomson-Haskell matrix method. The method has been used by several authors, e.g. Dorman (1962), Harkrider and Anderson (1962), Hannon (1964 a, b). Haskell (1960) applied this method to study the crustal reflection of plane SH waves. He derived an equation for the amplitude of the free surface displacement due to plane SH waves incident at any angle at the base of a layered crust. The same author (Haskell 1962) used this method for the crustal reflection of P- and SV-waves. He derived expressions, in terms of layer matrices for the motion of the free surface when plane harmonic P— and SV— waves are incident at any given angle at the base of a horizontally layered crust. In these investigations Haskell has shown the variation of the surface amplitude with the angle of incidence and time period for a single crustal layer model.

In the present attempt, we discuss the reflection of plane SH waves from a crust composed of transversely isotropic, homogeneous layers. The results of Haskell (1960) are obtained as a special case. Variation of the surface amplitude with the angle of incidence and frequency is shown graphically for two and three layered models.

Formulation of the problem

The medium is considered to be a layered half-space, composed of n parallel, homogeneous transversely isotropic, plane layers, the n^{th} layer being a homogeneous, transversely isotropic half-space. The numbering of layers and interfaces is shown in Fig. 1 (a). The rectangular cartesian co-ordinates have been taken with xy-plane in the free surface. The z-coordinate is measured normally into the medium. Associated with the m^{th} layer are its density ρ_m and directional rigidities (vertical and horizontal) M_m and N_m, respectively. We assume that the layer having thickness d_m is bounded by $z = z_{m-1}$ and $z = z_m$. The vertical and horizontal wave velocities βv_m and βh_m are given as

---

1 Paper No. 4 presented at Kurukshetra Symposium

2 Department of Mathematics, Kurukshetra University, Kurukshetra-132119, India.

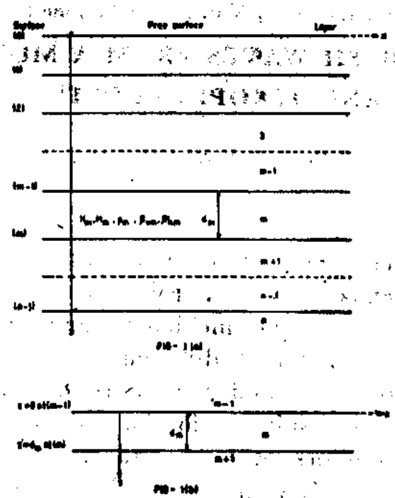

Fig. 1 Geometry of the problem

$$\left.\begin{aligned}\beta v_m &= \sqrt{\frac{M_m}{\rho_m}},\\ \beta h_m &= \sqrt{\frac{N_m}{\rho_m}},\end{aligned}\right\} \tag{1}$$

The equation of SH type wave motion in the mth layer is given by

$$\frac{\rho_m}{M_m}\ \frac{\partial^2 v_m}{\partial t^2} = \frac{N_m}{M_m}\ \frac{\partial^2 v_m}{\partial x^2} + \frac{\partial^2 v_m}{\partial z^2}, \tag{2}$$

where v_m stands for the y-component of the displacement. Equation (2) gives the displacement and tangential stress as under

$$\left.\begin{aligned} v_m(z) &= (A_m\, e^{i s_m z} + B_m\, e^{-i s_m z})\, e^{i(\omega t - ax)},\\ \tau_m(z) &= i\, M_m\, s_m (A_m\, e^{i s_m z} - B_m\, e^{-i s_m z})\, e^{i(\omega t - ax)},\end{aligned}\right\} \tag{3}$$

where ω is the angular frequency and

$$\left.\begin{aligned} s_m &= \left(\frac{\omega^2}{\beta^2 v_m} - a^2 \frac{N_m}{M_m}\right)^{1/2} \text{for } a\,(N_m/M_m)^{1/2} < \frac{\omega}{\beta v_m},\\ &= -\,i\left(a^2 \frac{N_m}{M_m} - \frac{\omega^2}{\beta^2 v_m}\right)^{1/2} \text{for } a\,(N_m/M_m)^{1/2} > \frac{\omega}{\beta v_m},\end{aligned}\right\} \tag{4}$$

A_m = displacement amplitude of the upward travelling SH wave,

B_m = displacement amplitude of the downward travelling SH wave,

$$a = \frac{\omega \sin \theta_m}{\beta h_m} \tag{5}$$

a is independent of m on account of Snell's law and θ_m is the angle which the wave makes with the positive direction of z-axis.

Reflection coefficient and surface amplitude

Let plane SH waves be incident from the half-space at the $(n-1)$ th interface at an angle θ_n. Let the plane of incidence be taken as xz-plane. We solve the problem subject to the following conditions :

(i) The free surface is stress free, that is,

$$\tau = 0 \quad \text{at} \quad z = 0 \tag{6}$$

(ii) The displacements and stresses are continuous at an interface. This implies

$$\left.\begin{aligned} v_i(z_i) &= v_{i+1}(z_i), \\ \tau_i(z_i) &= \tau_{i+1}(z_i), \qquad i=1, 2 \ldots n-1 \end{aligned}\right\} \tag{7}$$

Expressing the exponential functions in (3), in trigonometric form and supressing the common factors $e^{i(\omega t-ax)}$, we get

$$\left.\begin{aligned} v_m(z) &= (A_m + B_m)\cos(s_m z) + i(A_m - B_m)\sin(s_m z), \\ \tau_m(z) &= -M_m s_m (A_m + B_m)\sin(s_m z) + i\,M_m s_m (A_m - B_m)\cos(S_m z). \end{aligned}\right\} \tag{8}$$

Shifting the origin, temporarily, to the $(m-1)$th interface as shown in Fig. 1 (b), we have $z = 0$ at the $(m-1)$th interface. We denote the displacement and stress at the mth interface by v_m and τ_m, respectively. Then for the $(m-1)$th interface, equation (8) together with the boundary conditions (7) give

$$(v_{m-1}, \tau_{m-1}) = E_m (A_m + B_m, A_m - B_m), \tag{9}$$

where

$$E_m = \begin{bmatrix} 1 & 0 \\ 0 & i\,M_m s_m \end{bmatrix} \tag{10}$$

Similarly, (8) gives the values of the displacement and stress at the m th interface by putting $z = d_m$.

$$(v_m, \tau_m) = D_m (A_m + B_m, A_m - B_m), \tag{11}$$

where

$$D_m = \begin{bmatrix} \cos(s_m d_m) & i\sin(s_m d_m) \\ -M_m s_m \sin(s_m d_m) & i\,M_m s_m \cos(s_m d_m) \end{bmatrix} \tag{12}$$

Equation (9) gives

$$(A_m + B_m, A_m - B_m) = E_m^{-1} (v_{m-1}, \tau_{m-1}). \tag{13}$$

From (11) and (13), we get

$$(v_m, \tau_m) = D_m E_m^{-1} (v_{m-1}, \tau_{m-1}), \tag{14}$$

where

$$E_m^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{i\,M_m s_m} \end{bmatrix} \tag{15}$$

is the inverse of E_m. Equation (14) can be written as

$$(v_m, \tau_m) = a_m (v_{m-1}, \tau_{m-1}), \tag{16}$$

where

$$a_m = D_m E_m^{-1}. \tag{17}$$

Replacing m by m−1 in (16), we get

$$(v_{m-1}, \tau_{m-1}) = a_{m-1} (v_{m-2}, \tau_{m-2}). \tag{18}$$

Continuing this process, we get

$$(v_{n-1}, \tau_{n-1}) = a_{n-1}\, a_{n-2} \ldots a_1 (v_0, \tau_0), \tag{19}$$

where $v_0 = v_1 (z = 0)$, $\tau_0 = \tau_1 (z = 0)$ are respectively, the displacement and tangential stress at the free surf ce. From equations (13) and (19) we obtain

$$(A_n + B_n, A_n - B_n) = E_n^{-1} a_{n-1}\, a_{n-2} \ldots a_1 (v_0, \tau_0). \tag{20}$$

Putting

$$J_n = E_n^{-1} a_{n-1}\, a_{n-2} \ldots a_1 \tag{21}$$

Equation (20) gives

$$(A_n + B_n, A_n - B_n) = J_n (v_0, \tau_0). \tag{22}$$

If we restrict ourselves to the case of free surface then the boundary condition (6) implies that $\tau = 0$ and, therefore,

$$(A_n + B_n, A_n - B_n) = J_n (v_0, 0) \tag{23}$$

From equation (23), we get

$$R = \frac{B_n}{A_n} = \frac{(J_n)_{11} - (J_n)_{21}}{(J_n)_{11} + (J_n)_{21}}, \tag{24}$$

$$\frac{v_0}{A_n} = \frac{2}{(J_n)_{11} + (J_n)_{21}}. \tag{25}$$

Equations (24) and (25) are the expressions for the reflection coefficient and surface amplitude, respectively.

From equations (15), (17) and (21) we note that the elements of the matrices J_n are of the form

$$J_n = \begin{bmatrix} R & R \\ I & I \end{bmatrix}, \tag{26}$$

where R indicates the real quantity (not, of course, the same in both the positions) and an I indicate an imaginary quantity. Hence it follows from (26) that $(J_n)_{11}$ is real and $(J_n)_{21}$ is imaginary. This makes the numerator in (24), the complex conjugate of its denomenator showing that the amplitudes of the incident and the reflected waves are equal in absolute value and differ only in phase, i.e. whole of the energy is reflected back from the free surface.

Special case I (n = 2).

When n = 2, i.e. for a single layer over a half-space, (21) gives

$$J_2 = E_2^{-1} a_1. \tag{27}$$

Using equations (12), (15) and (17) in (27), we have

$$J_2 = \begin{bmatrix} \cos Q_1 & \dfrac{\sin Q_1}{M_1 s_1} \\ -\dfrac{M_1 s_1 \sin Q_1}{M_2 s_2} & -\dfrac{\cos Q_1}{i M_2 s_2} \end{bmatrix} \tag{28}$$

where $Q_m = s_m d_m$. From equation (24), (25) and (28), we get

$$R = \frac{B_2}{A_2} = \frac{\cos Q_1 - i \dfrac{M_1 s_1}{M_2 s_2} \sin Q_1}{\cos Q_1 + i \dfrac{M_1 s_1}{M_2 s_2} \sin Q_1}, \tag{29}$$

$$\frac{v_0}{A_2} = \frac{2}{\cos Q_1 + i \dfrac{M_1 s_1}{M_2 s_2} \sin Q_1}, \tag{30}$$

where from (4) and (5)

$$s_1 = \frac{\omega}{\beta h_2}\left[\frac{N_1}{M_1}\left(\frac{\beta h_2}{\beta^2 h_1} - \sin^2 \theta_2\right)\right]^{1/2}, \text{ for } \sin \theta_2 < \frac{\beta h_2}{\beta h_1}$$

$$= -i \frac{\omega}{\beta h_2}\left[\frac{N_1}{M_1}\left(\sin^2 \theta_2 - \frac{\beta^2 h_2}{\beta^2 h_1}\right)\right]^{1/2}, \text{ for } \sin \theta_2 > \frac{\beta h_2}{\beta h_1}$$

If anisotropy vanishes, we have

$$\left.\begin{aligned} N_i &= M_i = u_i \text{ (say)}, \\ \beta h_i &= \beta v_i = \beta_i \text{ (say)}, \\ i &= 1, 2 \end{aligned}\right\} \tag{31}$$

Then, equation (29) gives

$$R = \frac{B_2}{A_2} = \frac{\cos Q_1 - \dfrac{u_1}{u_2}\left(\dfrac{\beta_2^2}{\beta_1^2} - \sin^2 \theta_2\right)^{1/2} \sin Q_2}{\cos Q_1 + \dfrac{u_1}{u_2}\left(\dfrac{\beta_2^2}{\beta_1^2} - \sin^2 \theta_2\right)^{1/2} \sin Q_2}. \tag{32}$$

By making use of Snell's law and after some simplification, we get

$$R = \frac{\cos Q_1 - i b \sin Q_1}{\cos Q_1 + i b \sin Q_1}, \tag{33}$$

where

$$\left.\begin{aligned} b(c) &= \frac{u_1}{u_2} \cdot \frac{r\beta_1}{r\beta_2}, \\ r\beta_i &= [(c/\beta_i)^2 - 1]^{1/2} \\ i &= 1, 2 \end{aligned}\right\} \tag{34}$$

In the equation (34) c is horizontal phase velocity. Equation (30) reduces to

$$\frac{v_0}{A_2} = \frac{2}{\cos Q_1 + i b \sin Q_1}. \tag{35}$$

Equations (33) and (35) are the expressions for the reflection coefficient and the surface amplitude respectively, when an isotropic, homogeneous layer is lying over an isotropic, homogeneous half-space (Haskell, 1960).

Special case II. (n = 3)

Using equations (24) and (25) and following the above procedure, we get value for the reflection coefficient and the surface amplitude as given below :

$$R = \frac{\left(\cos Q_1 \cos Q_2 - \frac{M_1 s_1}{M_2 s_2} \sin Q_1 \sin Q_2\right) - i\left(\frac{M_2 s_2}{M_3 s_3} \sin Q_2 \cos Q_1 + \frac{M_1 s_1}{M_3 s_3} \sin Q_1 \cos Q_2\right)}{\left(\cos Q_1 \cos Q_2 - \frac{M_1 s_1}{M_2 s_2} \sin Q_1 \sin Q_2\right) + i\left(\frac{M_2 s_2}{M_3 s_3} \sin Q_2 \cos Q_1 + \frac{M_1 s_1}{M_3 s_3} \sin Q_1 \cos Q_2\right)} \tag{36}$$

$$\frac{v_0}{A_3} = \frac{2}{\left(\cos Q_1 \cos Q_2 - \frac{M_1 s_1}{M_2 s_2} \sin Q_1 \sin Q_2\right) + i\left(\frac{M_2 s_2}{M_3 s_3} \sin Q_2 \cos Q_1 + \frac{M_1 s_1}{M_3 s_3} \sin Q_1 \cos Q_2\right)} \tag{37}$$

Numerical Calculations

Numerical calculations have been carried out for single and double layer cases. Equation (29) gives the reflection coefficient for a single layer case, for $\frac{\beta h_2}{\beta h_1} > \sin\theta_2$ on the other hand, when $\frac{\beta h_2}{\beta h_1} < \sin\theta_2$, we find

$$R = \frac{\cosh \bar{Q}_1 + i\,\frac{M_1 \bar{s}_1}{M_2 s_2} \sinh \bar{Q}_1}{\cosh \bar{Q}_1 - i\,\frac{M_1 \bar{s}_1}{M_2 s_2} \sinh \bar{Q}_1}, \tag{38}$$

where

$$\bar{s}_1 = \frac{\omega}{\beta h_2}\left[\frac{N_1}{M_1}\left(\sin^2\theta_2 - \frac{\beta^2 h_2}{\beta^2 h_1}\right)\right]^{1/2},$$

and

$$\bar{Q}_1 = \bar{s}_1 d_1 \tag{39}$$

The corresponding value for the surface amplitude is

$$\frac{v_0}{A_2} = \frac{2}{\cos h\,\bar{Q}_1 - i\,\frac{M_1 \bar{s}_1}{M_2 s_2} \sin h\,\bar{Q}_1}. \tag{40}$$

From equations (29) and (38) we note that the value of the modulus of the reflection coefficient is always equal to unity. The surface amplitude is given by (30) and (40). The values of the modulus of the surface amplitude have been computed. We have chosen $\frac{\beta h_2}{\beta h_1} = 1.3$. The anisotropy factors of both the media are given two sets of values each, namely $\frac{N_1}{M_1} = 0.6$, 1.2 and $\frac{N_2}{M_2} = 0.8$, 1.4. The values of the surface amplitude have been plotted against the angle of incidence and the dimensionless frequency $\Omega = \frac{\omega d_1}{\beta h_1}$. Fig. 2 shows the variation of the modulus of the surface amplitude with the angle of incidence when $\frac{N_1}{M_1} = 0.6$ and $\frac{N_2}{M_2} = 0.8$. Three curves in this figure correspond to three different values given to Ω, namely $\Omega = 2, 10, 18$. Similarly in the Fig. 3 the variation of the surface amplitude with the angle of incidence is shown for $\frac{N_1}{M_1} = 1.2$ and $\frac{N_2}{M_2} = 1.4$. Fig 4 shows

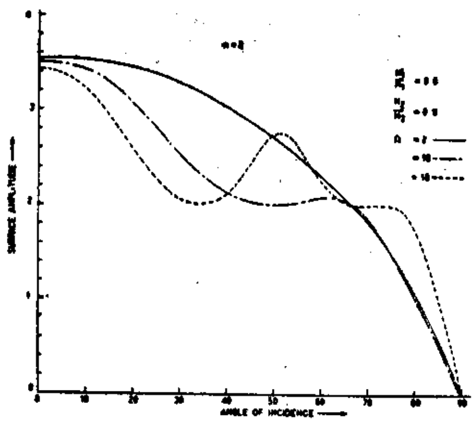

Fig. 2 The variation of the surface amplitude with the angle of incidence for $N_1/M_1 = 0.6$ and $N_2/M_2 = 0.8$

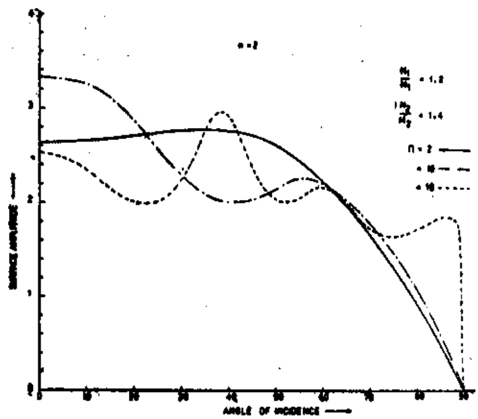

Fig. 3 The variation of the surface amplitude with the angle of incidence for $N_1/M_1 = 1.2$ and $N_2/M_2 = 1.4$

the variation of the absolute value of the surface amplitude with Ω for the first set of values given to anisotropy factors. One curve corresponds to the normal incidence and the other curve is for 30° angle of incidence. For second set of values of anisotropy factors, the variation of the surface amplitude with Ω is shown in Fig. 5. Fig. 6 compares the surface amplitude for the two cases of anisotropy and one case of isotropy. A similar comparision of the variation of the surface amplitude with Ω is shown in Fig. 7. As can be seen from the graphs obtained, the surface amplitude changes considerably as the value of anisotropy factors are changed.

For two layer case we note that the equations (36) and (37) hold for $\frac{\beta h_2}{\beta h_1} > \sin\ \theta_3$ and $\frac{\beta h_3}{\beta h_2} > \sin\theta_3$. Corresponding equations for the cases

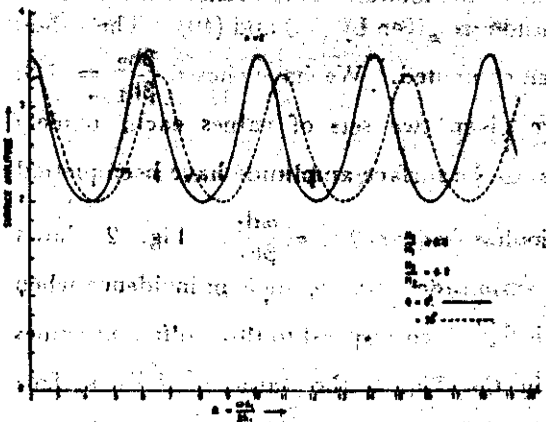

Fig. 4 The variation of the surface amplitude with the frequency for $N_1/M_1 = 0.6$ and $N_2/M_2 = 0.8$

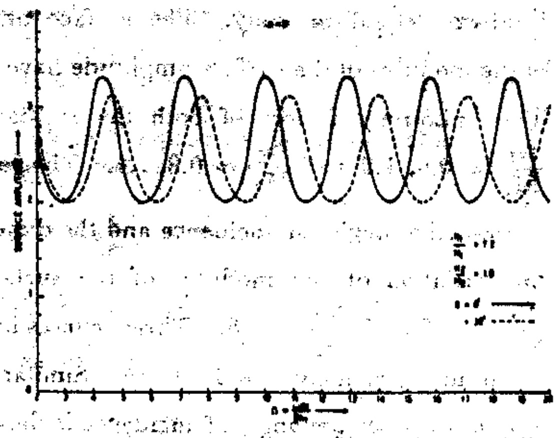

Fig. 5 The variation of the surface amplitude with the frequency for $N_1/M_1 = 1.2$ and $N_2/M_2 = 1.4$

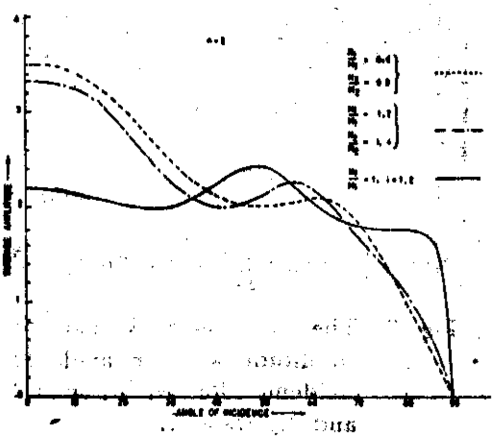

Fig. 6 Comparision of the variation of the surface amplitude with the angle of incidence between two cases of and anisotropy one case of isotropy for $\Omega = 10$

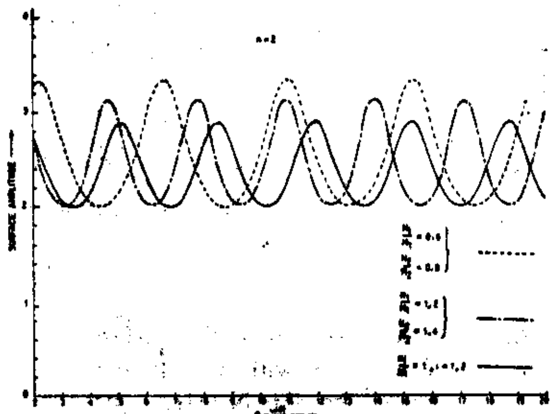

Fig. 7 Comparision of the variation of the surface amplitude with the frequency between two cases of anisotropy and one case of isotropy for 30° angle of incidence

(i) $\frac{\beta h_3}{\beta h_1} < \sin\theta_3,\ \frac{\beta h_3}{\beta h_2} > \sin\theta_3$

(ii) $\frac{\beta h_3}{\beta h_1} > \sin\theta_3,\ \frac{\beta h_3}{\beta h_2} < \sin\theta_3$

(iii) $\frac{\beta h_3}{\beta h_1} < \sin\theta_3,\ \frac{\beta h_3}{\beta h_2} < \sin\theta_3.$

can be obtained on modifying the radicals suitably. We note that absolute value of the reflection coefficient is unity in all the cases which being the same as required by the energy conservation. Absolute values of the surface amplitude can be calculated from equation (37) or its modified forms. For numerical purposes, we have chosen $\frac{\beta h_3}{\beta h_1} = 1.3$ and $\frac{\beta h_2}{\beta h_1} = 1.24$. The ratio of the thickness of the layers, i e. $\frac{d_2}{d_1}$ is taken to be 0.8. The anisotropy factors are given different values, namely $\frac{N_1}{M_1} = 0.6$, $\frac{N_2}{M_2} = 0.7$, $\frac{N_3}{M_3} = 0.8$ and $\frac{M_3}{M_1} = 2.0$, $\frac{M_2}{M_1} = 1.8$. Fig. 8 shows the variation of the surface amplitude with the angle of incidence for $\Omega = 2$, 10 and 18. The variation of the modulus of the surface amplitude with frequency is shown in Fig. 9.

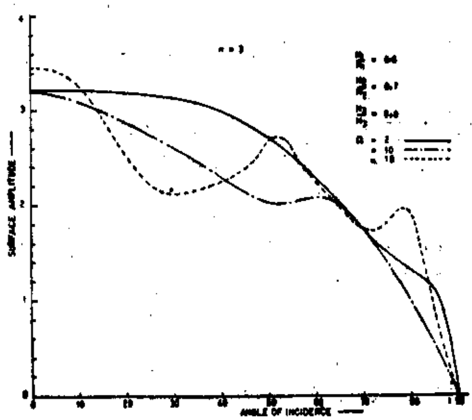

Fig. 8 The variation of the surface ampliude with the angle of incidence for $N_1/M_1 = 0.6$, $N_2/M_2 = 0.7$ and $N_3/M_3 = 0.8$

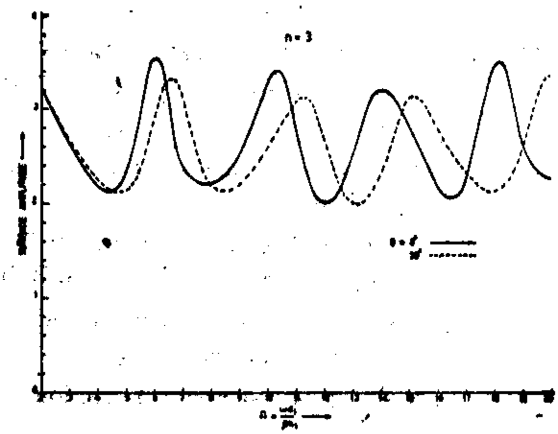

Fig. 9 The variation of the surface amplitude with the frequcncy for $N_1/M_1 = 0.6$, $N_2/M_2 = 0.7$ and $N_3/M_3 = 0.8$

Conclusions

The modulus of the surface amplitude has been plotted against angle of incidence and frequency. Comparision of variation of the surface amplitude with the angle of incidence and frequency has been made for three cases, namely two cases of anisotropy and one case of isotropy. It is noted that absolute value of the surface amplitude is zero for 90° angle of incidence in every case. It is concluded from the above analysis that the anisotropy of the medium produces a marked effect on the surface amplitude.

Acknowledgements

This research was financed by a Research Fellowship from University Grants Commission. The Computational work was performed on the Electronic Computer IDC-316 at the Computer Centre, Kurukshetra University, Kurukshetra. The author is indebted to Prof.

S. J. Singh, Head, Department of Mathematics, M.D. University, Rohtak for suggesting the problem and helping throughout the preparation of the work. The author is also thankful to Prof. C. Mohan, Head, Department of Mathematics, Kurukshetra University, Kurukshetra for his keen interest and constant encouragement.

References

Anderson, D.L. (1962)—Love wave dispersion in heterogeneous anisotropic media, Geophysics 27, pp. 445-454.

Brekhovskikh, L.M. (1960)—Love waves in layered media, Academic Press, New York.

Dorman, J. (1962)—Period equation for wave of Rayleigh type on a multilayered, liquid solid half-space, Bull. Seism. Soc. Amer., 52, pp. 389-397.

Ewing, W.M., Jardetzky, W.S. and Press, F. (1957)—Elastic waves in layered media, McGraw-Hill, New York.

Hannon. W.J. (1964 a)—Techn. Report A F. Cambridge Research Laboratory, Bedford, Mass. Rept. No. AFCRL, pp. 1564-1614.

Hannon, W.J. (1964 b)—An application of Thomson-Haskell matrix method to the synthesis of the surface motion due to dilatational waves, Bull. Seism. Soc. Amer., 54, pp. 2067-2070.

Harkrider, D.G. and Anderson, D.L. (1962)—Computation of surface wave dispersion for multilayered anisotropic media, Bull. Seism. Soc. Amer., 52, pp. 321-332.

Haskell, N.A. (1953)—The dispersion of surface waves on multilayered media, Bull. Seism. Soc. Amer., 43, pp. 17-34.

Haskell, N.A. (1960)—Crustal reflection of plane SH waves, J. Geophys. Res., 65, pp. 4147-4150.

Haskell, N.A. (1962)—Crustal reflection of plane P and SV waves, J. Geophys. Res., 67, pp 4751-4767.

Thomson, W.T. (1950)—Transmission of elastic waves through a stratified solid medium, J. Appl. Phy., 21, pp. 89-93.