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FREE VIBRATION OF ELASTIC MEDIUM

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INTRODUCTION

The dynamics of elastic media has been studied in the past mainly from the wave propagation point of view (Ewing *et al*, 1957). This is probably because of the importance of such problems in fields like geophysics, wherein one mostly considers layers of infinite length or layers of infinite depth. However, while considering finite elastic media the approach of the theory of vibrations would be more informative and useful. As an example the inplane vibrations of a gravity dam idealized as a thin rectangular medium can be cited. Also in dynamic soil-structure interaction studies the foundation may sometimes be modelled as a two-dimensional elastic medium of finite depth and width, resting on a rock bed. In such problems probably the most desired results are the natural frequncies and mode shapes for the free vibration of the two-dimensional medium. These can be used invariably in forced vibration studies. These modelling of soil deposits as elastic continua for purposes of dynamic analysis is not new. Previously Idriss (1968) and Idriss and Seed (1967, 1968, 1970) while studying the seismic response of soil layers, have suggested an one-dimensional analysis for semi-infinite layers. Clough and Chopra (1966) have analysed an earth dam section as a two-dimensional elastic medium by the finite element method. Eernisse (1966) has considered two-dimensional problems in dynamic elasticity for which closed form solutions are generally not available. Exact solutions may not be possible because of the complicated boundary conditions that are to be satisfied at the edges. Nevertheless, general expressions for the eigenfunctions would be most desirable since the forced response can be expanded as a series in terms of these functions.

Herein, the natural frequencies and mode shapes in closed form are found for a finite elastic medium under two typical sets of boundary conditions. The medium is assumed to be held at the lower edge against displacements and is taken to be stress free at the top. These are infact the conditions met in practice also. For the lateral edges two variants in the conditions are used. An orthogonality relationship of the modes is also presented in the sequel.

TWO-DIMENSIONAL MEDIUM

The system under consideration is a rectangular elastic medium as shown in Figure 1. For the inplane vibrations of the medium the equations of motion are (Ewing *et al*, 1957)

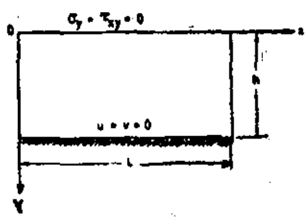

Fig. 1. The two-dimensional elastic-medium.

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}+(\lambda+\mu)\frac{\partial^2 v}{\partial x \partial y}=m\frac{\partial^2 u}{\partial t^2} \quad \ldots(1)$$

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$$(\lambda+2\mu)\frac{\partial^2 v}{\partial y^2}+\mu\frac{\partial^2 v}{\partial x^2}+(\lambda+\mu)\frac{\partial^2 u}{\partial x\partial y}=m\frac{\partial^2 v}{\partial t^2} \quad \ldots(2)$$

Here, $u(x, y, t)$ and $v(x, y, t)$ are the displacements in the X- and Y- directions respectively. λ, μ and m are the bulk modulus, shear modulus and mass density of the medium.

The most useful set of boundary conditions for the top and bottom edges are as follows:

$$\sigma_y(x, 0, t)=\tau_{xy}(x, 0, t)=0 \quad \ldots(3)$$

$$u(x, h, t)=v(x, h, t)=0 \quad \ldots(4)$$

Here, σ_y and τ_{xy} are the normal and shear stresses given by

$$\sigma_y=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial v}{\partial y} \quad \ldots(5)$$

$$\tau_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \quad \ldots(6)$$

For the other two edges the following two sets of conditions are considered:

Case 1:

$$u(0, y, t)=u(L, y, t)=0 \quad \ldots(7)$$

$$\partial v(0, y, t)/\partial x=\partial v(L, y, t)/\partial x=0 \quad \ldots(8)$$

Case 2:

$$v(0, y, t)=v(L, y, t)=0 \quad \ldots(9)$$

$$\partial u(0, y, t)/\partial x=\partial u(L, y, t)/\partial x=0 \quad \ldots(10)$$

These are believed to be realistic since in practice the late-al conditions are uncertain and probably lie in between the strict stress-free and the displacement-free conditions.

ANALYSIS

The free vibration solution to equations (1) and (2) can be taken in the form

$$u(x, y, t)=U(x, y)\sin\omega t \quad \ldots(11)$$

$$v(x, y, t)=V(x, y)\sin\omega t \quad \ldots(12)$$

where ω is the circular natural frequency of the medium. The resulting coupled equations for U and V can be uncoupled by introducing the displacement potentials Φ and Ψ defined as

$$U=\frac{\partial\Phi}{\partial x}+\frac{\partial\Psi}{\partial y} \quad \ldots(13)$$

$$V=\frac{\partial\Phi}{\partial y}-\frac{\partial\Psi}{\partial x} \quad \ldots(14)$$

The equations for Φ and Ψ would be

$$\nabla^2\Phi=-[m\omega^2/(\lambda+2\mu)]\,\Phi \quad \ldots(15)$$

$$\nabla^2\Psi=-[m\omega^2/\mu]\,\Psi \quad \ldots(16)$$

Since the solutions for both the cases mentioned above are similar, case 1 will be considered first in detail. The solutions for Φ and Ψ which satisfy the boundary conditions given by equations (7) and (8) can be chosen in the form

$$\Phi_{ij}(x, y)=\phi_{ij}(y)\cos(i\pi x/L) \quad \ldots(17)$$

$$\Psi_{ij}(x, y)=\psi_{ij}(y)\sin(i\pi x/L) \quad \ldots(18)$$

where $i=0, 1, 2, \ldots$ and $j=1, 2, 3, \ldots$.
Substitution of these expressions in equations (15) and (16) leads to

$$\phi_{ij}''+(\alpha^2/h^2)\,\phi_{ij}=0 \qquad \ldots(19)$$

$$\psi_{ij}''+(\beta^2/h^2)\,\psi_{ij}=0 \qquad \ldots(20)$$

Here, primes denote differentiation with respect to y, and

$$\alpha^2=m\omega^2h^2/(\lambda+2\mu)-(i\pi h/L)^2 \qquad \ldots(21)$$

$$\beta^2=m\omega^2h^2/\mu-(i\pi h/L)^2 \qquad \ldots(22)$$

Since ω^2 is positive, α and β have to satisfy the condition

$$\beta^2-\alpha^2\geqslant 0 \qquad \ldots(23)$$

Also, since the Poisson's ratio cannot be greater than 0.5, α^2 and β^2 cannot vanish simultaneously. Due to these constraints, only five combinations of α and β are possible. The solutions of equations (19) and (20) in the various cases along with the boundary conditions given by equations (3) and (4) lead to the following frequency equations.

Case I $i=0$

In this case both α^2 and β^2 are positive. The frequency equation is

$$\cos\alpha=0 \qquad \ldots(24)$$

the roots of which are given by

$$\alpha_{0j}=(2j-1)\,\pi/2 \qquad \ldots(25)$$

These roots give the solution for the displacements are

$$U_{0j}(x, y)=0$$

$$V_{0j}(x, y)=(\alpha_{0j}/h)\cos(\alpha_{0j}y/h) \qquad \ldots(26)$$

A non-dimensional frequency parameter Ω can be conveniently defined in the form

$$\Omega_{ij}^2=m\omega_{ij}^2h^2/\mu \qquad \ldots(27)$$

Now, from equations (21) and (25) it follows

$$\Omega_{0j}=(2+\lambda/\mu)^{1/2}\,(j-\tfrac{1}{2})\,\pi \qquad \ldots(28)$$

Case I $i\neq 0,\ \alpha^2>0,\ \beta^2>0$:

For this case, the transcendental equation leading to the natural frequencies is

$$4-(4\alpha\beta/\Delta+\Delta/\alpha\beta)\sin\alpha\sin\beta+(4\gamma^2/\Delta+\Delta/\gamma^2)\cos\alpha\cos\beta=0 \qquad \ldots(29)$$

where

$$\gamma=(i\pi h/L) \qquad \ldots(30)$$

and

$$\Delta=\beta^2-\gamma^2 \qquad \ldots(31)$$

The eigenfunctions for the displacements in the X- and Y- directions are

$$U_{ij}(x, y)=[(2\alpha\beta\gamma/h\Delta)\sin(\beta y/h)-(\gamma/h)\sin(\alpha y/h)-C_1(\Delta/2\gamma h)\cos(\beta y/h)$$
$$-C_1(\gamma/h)\cos(\alpha y/h)]\sin(i\pi x/L) \qquad \ldots(32)$$

$$V_{ij}(x, y)=[(\alpha/h)\cos(\alpha/h)+(2\alpha\gamma^2/h\Delta)\cos(\beta y/h)-C_1(\alpha/h)\sin(\alpha y/h)$$
$$+C_1(\Delta/2\beta h)\sin(\beta y/h)]\cos(i\pi x/L) \qquad \ldots(33)$$

where

$$C_1=[(2\alpha\beta/\Delta)\sin\beta-\sin\alpha]/[(\Delta/2\gamma^2)\cos\beta+\cos\alpha] \qquad \ldots(34)$$

Case I $i\neq 0,\ \alpha^2<0,\ \beta^2>0$:

The transcendental equation for this case is

$$4+(4\Delta_1\beta/\Delta)-\Delta/\Delta_1\beta)\sinh\Delta_1\sin\beta+(4\gamma^2/\Delta+\Delta/\gamma^2)\cosh\Delta_1\cos\beta=0 \qquad \ldots(35)$$

where

$$\Delta_1=(|\alpha^2|)^{1/2} \qquad \ldots(36)$$

The eigenfunctions are given by

$$U_{ij}(x, y)=[(2\beta\Delta_1\gamma/4\Delta)\sin(\beta y/h)-(\gamma/h)\sinh(\Delta_1 y/h)-C_2(\Delta/2\gamma h)\cos(\beta y/h)$$
$$-C_2(\gamma/h)\cosh(\Delta_1 y/h)]\sin(i\pi x/L) \quad \ldots(37)$$

$$V_{ij}(x, y)=[(\Delta_1/h)\cosh(\Delta_1 y/h)+(2\Delta_1\gamma^2/h\Delta)\cos(\beta y/h)+C_2(\Delta_1/h)\sinh(\Delta_1 y/h)$$
$$+C_2(\Delta/2\beta h)\sin(\beta y/h)]\cos(i\pi x/L) \quad \ldots(38)$$

where $C_2=[(2\Delta_1\beta/\Delta)\sinh\beta-\sinh\Delta_1]/[(\Delta/2\gamma^2)\cos\beta+\cosh\Delta_1]$ ...(39)

Case I $i\neq 0,\ \alpha^2<0,\ \beta^2<0$:

For this case the equation leading to the natural frequencies is

$$4-(4\Delta_1\Delta_2/\Delta+\Delta/\Delta_1\Delta_2)\sinh\Delta_1\sinh\Delta_2+(4\gamma^2/\Delta+\Delta/\gamma^2)\cosh\Delta_1\cosh\Delta_2=0 \quad \ldots(40)$$

where $\Delta_2=(|\beta^2|)^{1/2}$. ...(41)

The mode shapes are

$$U_{ij}(x, y)=[C_3(\Delta/2\gamma\Delta_2)\cosh(\Delta_2 y/h)+C_3(\gamma/h)\cosh(\Delta_1 y/h)$$
$$-(2\Delta_1\Delta_2\gamma/\Delta)\sinh(\Delta_2 y/h)-(\gamma/h)\sinh(\Delta_1 y/h)]\sin(i\pi\gamma/L) \quad \ldots(42)$$

$$V_{ij}(x, y)=[(\Delta_1/h)\cosh(\Delta_1 y/h)+(2\Delta_1\gamma^2/h\Delta)\cosh(\Delta_2 y/h)$$
$$-C_3(\Delta_1/h)\sinh(\Delta_1 y/h)-C_3(\Delta/2\Delta_2\gamma)\sinh(\Delta_2 y/h)]\cos(i\pi x/L) \quad \ldots(43)$$

where $C_3=[(2\Delta_1\Delta_2\gamma/\Delta)\sinh\Delta_2+\sinh\Delta_1]/[(\Delta/2\gamma^2)\cosh\Delta_2+\cosh\Delta_1]$ ...(44)

Case I $i\neq 0,\ \alpha^2=0,\ \beta^2>0$:

The frequency equation is

$$4-[(\lambda/\mu)\beta/(1+\lambda/\mu)]\sin\beta+(4\mu/\lambda+\lambda/\mu)\cos\beta=0 \quad \ldots(45)$$

The eigenfunctions are

$$U_{ij}(x, y)=[2C_4(\mu/\lambda)(\beta/\gamma)\sin(\beta y/h)-\gamma C_4(y/h)-(1/2)(\lambda/\mu)(\gamma/h)\cos(\beta y/h)-(\gamma/h)]\sin i\pi x/L) \quad \ldots(46)$$

$$V_{ij}(x, y)=[C_4+2C_4(\mu/\lambda)\cos(\beta y/h)-(\lambda/2\mu)(\gamma^2/h\beta)\sin(\beta y/h)]\cos(i\pi x/L) \quad \ldots(47)$$

where $C_4=[(\lambda/2\mu)(\gamma^2/4\beta)\sin\beta]/[1+2(\mu/\lambda)\cos\beta]$ ...(48)

Case I $i\neq 0,\ \alpha^2<0,\ \beta^2=0$:

The frequency equation for this case is obtained as

$$4+[\Delta_1(2+\lambda/\mu)/(1+\lambda/\mu)]\sinh\Delta_1-5\cosh\Delta_1=0 \quad \ldots(49)$$

The eigenfunctions are

$$U_{ij}(x, y)=[(2/\gamma h)+C_5(\gamma/h)\cosh(\Delta_1 y/h)-(\gamma/h)\sinh(\Delta_1 y/h)]\sin(i\pi x/L) \quad \ldots(50)$$

$$V_{ij}(x, y)=[(\Delta_1/h)\cosh(\Delta_1 y/h)-C_5(\Delta_1/h)\sinh(\Delta_1 y/h)-2(\Delta_1/h)-(2/h)(y/h)]\cos(i\pi x/L) \quad \ldots(51)$$

where $C_5=\sinh\Delta_1/[\cosh\Delta_1-2/\gamma^2]$ ...(52)

SOLUTION WITH BOUNDARY CONDITIONS OF CASE 2

The solutions for Φ and Ψ which satisfy the boundary conditions of equations (9) and (10) can be chosen in the form

$$\Phi_{ij}(x, y)=\phi_{ij}(y)\sin(i\pi x/L) \quad \ldots(53)$$

$$\Psi_{ij}(x, y)=\psi_{ij}(y)\cos(i\pi x/L) \quad \ldots(54)$$

The analysis is very similar to that of the first case. Only the final results are presented here.

1 Case 2, i=0:

The frequency equation is

$$\cos\beta=0 \quad \ldots(55)$$

This yields

$$\beta_{0j}=\Omega_{0j}=(j-\tfrac{1}{2})\,\pi \qquad \ldots(56)$$

and

$$U_{0j}\,(x,\,y)=-(\beta_{0j}/h)\cos\,(\beta_{0j}\,y/h) \qquad \ldots(57)$$

$$V_{0j}(x,\,y)=0 \qquad \ldots(58)$$

2 Case 2, $i\neq 0$:

For $i=1, 2, 3, \ldots$, the same combinations of α^2 and β^2 as considered for the first set of boundary conditions are to be considered here also. It is interesting to note that the frequency equations remain the same as before. The eigenfunctions may be obtained from the previous solutions using the following simple relations.

$$U_{ij}\,(x,\,y)_2/\cos\,(i\pi x/L)=-U_{ij}\,(x,\,y)_1/\sin\,(i\pi x/L) \qquad \ldots(59)$$

$$V_{ij}\,(x,\,y)_2/\sin\,(i\pi x/L)=V_{ij}\,(x,\,y)_1/\cos\,(i\pi x/L) \qquad \ldots(60)$$

Here the subscripts 1 and 2 refer to case 1 and case 2 respectively.

ORTHOGONALITY CONDITION

For the present problem the eigenfunctions satisfy an orthogonality condition of the type

$$\int_0^h\int_0^L U_{ij}\,(x,\,y)\,U_{kl}\,(x,\,y)+V_{ij}\,(x,\,y)\,V_{kl}(x,\,y)dx\,dy=0 \qquad \ldots(61)$$

for $i\neq k$ or $j\neq l$

Since the eigenfunctions are of the form

$$U_{ij}\,(x,\,y)=U_{ij}(y)\,\sin\,i\pi x/L \qquad \ldots(62)$$

$$V_{ij}\,(x,\,y)=V_{ij}\,(y)\,\cos\,i\pi x/L \qquad \ldots(63)$$

a further orthogonality relation of the form

$$\int^h (U_{ij}\,U_{ik}+V_{ij}\,V_{ik})\,dy=0 \qquad \ldots(64)$$

is valid for every i and $j\neq k$. The proofs of these assertions are very straight forward and hence they have been omitted here.

NUMERICAL RESULTS

The various transcendental equations leading to the natural frequencies have been solved numerically on a digital computer. Detailed results have been obtained for the frequency parameter Ω^2_{ij} over a wide range of values of ih/L. Five values of λ/μ, namely, 0, 2/3, 1, 2 and 9 have been chosen for the numerical work. The results are presented in figures 2 through 6. The last value of λ/μ, namely, 9 corresponds to a Poisson's ratio value of 0.45. [This is supposed to representative of a soil medium]. The frequencies corresponding to the modes associated with $i=0$ have also been shown in these figures.

DISCUSSION AND CONCLUSIONS

The analysis shows that for both the types of boundary conditions the natural frequencies for $i\neq 0$ are governed by the same transcendental equations although the eigenfunctions are different. However, the ordering of the natural frequencies would be different, for the two cases. This can be seen from figures 2 through 6. The fundamental frequency for the case 2 type of boundary conditions is governed by the $i=0$ mode, whereas this is not true for the other type of boundary condition.

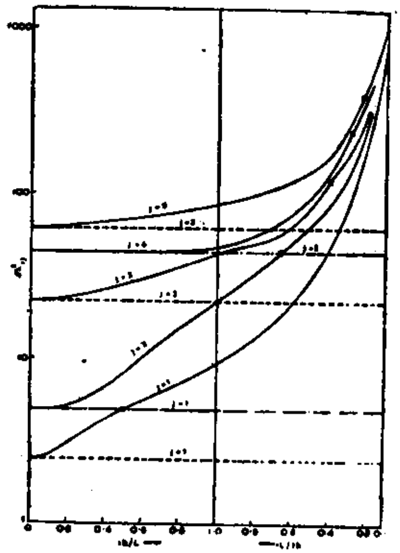

Fig. 2. Variation of Natural Frequency;
Poisson's ratio=0; λ/μ=0; — equations 29, 35, 40;
□ equation 45; —·—· equation 28; — — —equation 56.

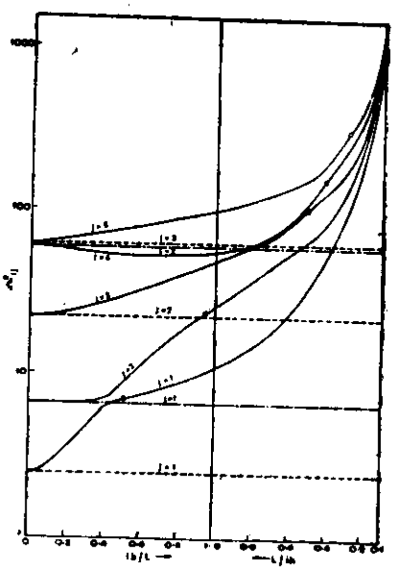

Fig. 3. Variation of Natural Frequency;
Poisson's ratio=0.2; λ/μ=2/3; — equations 29, 35, 40;
□ equation 45; —·—·equation 28; — — —equation 56.

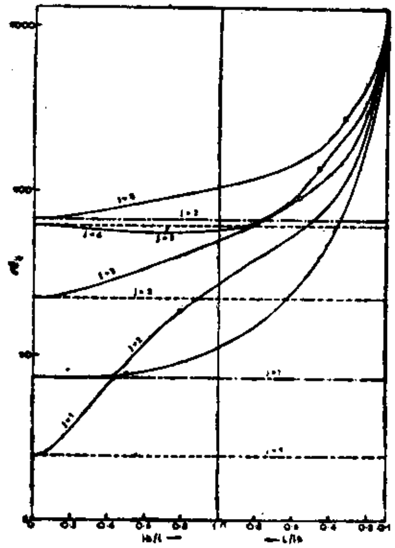

Fig. 4. Variation of Natural Frequency;
Poisson's ratio=0.25; λ/μ=1.0; — equations 29, 35, 40;
□ equation 45; —·—· equation 28; — — — equation 56.

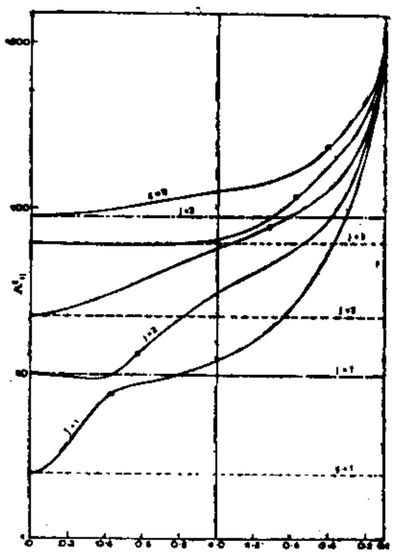

Fig. 5. Variation of Natural Frequency;
Poisson's ratio=1/3; λ/μ=2.0; — equations 29, 35, 40;
□ equation 45; —·—·— equation 28; — — — equation 56.

The non-dimensional parameter h/L always occurs in the form ih/L. With this in view the results have been shown against ih/L and L/ih. As the parameter ih/L tends to zero, all the natural frequencies converge to the values corresponding to $i=0$.

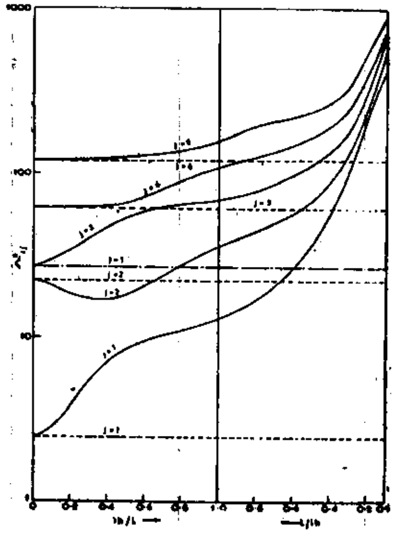

Fig 6. Variation of Natural Frequency;
Poisson's ratio=0.45; $\lambda/\mu=9.0$; — equations 29, 35, 40;
□equation 45; —•—•—equation 28; — — —equation 56.

An interesting observation in this connection is that for the boundary conditions of the second type given by equations (9) and (10) the natural frequencies corresponding tc $i=0$ are same as the frequencies of a semi-infinite layer as studied by Idriss and Seed (1968) using an one dimensional theory.

Apart from leading to the natural frequencies the free vibration studies are of immense importance in forced vibration analysis. In earthquake engineering problems, the external excitation appears generally in the form of a time dependent boundary condition. For the two-dimensional medium considered herein, a useful result would be the response and stress distribution for earthquake excitations at the edge $y=h$. Such a study is in progress at present.

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