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SURFACE WAVE PROPAGATION IN A HOMOGENEOUS ANISOTROPIC LAYER LYING OVER A HOMOGENEOUS SEMI ISOTROPIC HALF-SPACE, AND UNDER A UNIFORM LAYER OF LIQUID[1]

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Introduction

In the study of surface waves we make certain drastic assumptions. Although the assumption of isotropy is often approximately satisfied in practice, certain disagreements exist between theory and observation that indicate the necessity of discussing problems of dispersion under less restrictive and possibly more realistic assumptions. In recent years the propagation of elastic waves in anisotropic media has begun to receive some attention. The assumption that the physical properties of an elastic medium vary with repect to direction at a fixed point in a medium characterises the medium as being anisotropic. There are reasonable grounds for the assumption that anisotropy may exist in the continents. An obvious consideration is that materials deposited in water may settle in preferred orientation.

Stoneley (1926), Biot (1952) and Tolstoy (1954) have considered the propagation of elastic waves in a system consisting of a liquid layer of finite depth overlying an isotropic half space. Stoneley (1957) considered the case of ocean layering. Abubaker and Hudson (1961) studied the dispersive properties of liquid overlying a semi-infinite homogeneous anisotropic (transversely isotropic) half space.

Here we have considered the problem (two-dimensional) of surface wave propagation in a homogeneous anisotropic layer lying over a homogeneous semi-infinite isotropic half space, and under a uniform layer of liquid. This appears to be of practical interest as the bottom of the ocean is likely to be anisotropic (transversely isotropic) to some kilometres depth since formed by deposited material. The frequency equation for progressive surface waves is obtained and phase and group velocities are calculated as functions of wave number, using the constants for beryl to represent the anisotropic layer. The following special cases have been deduced.

(i) By making the depth of the liquid layer zero, the frequency equation for Rayleigh type waves in a transversely isotropic layer lying over an isotropic half space has been derived.

(ii) By making the depth of the transversely isotropic layer zero, the frequency equation obtained by Tolstoy (1954) has been derived.

Stresses and displacements

We consider a medium consisting of a liquid layer of thickness h, density ρ_1 and bulk modulus λ_1 lying over a layer of homogeneous anisotropic material of thickness h′ with

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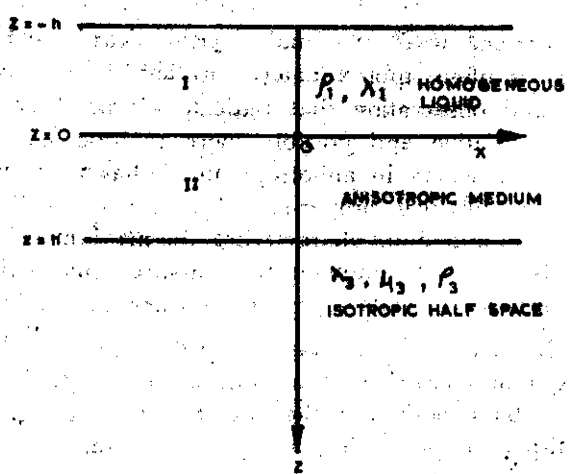
density ρ_2 whose elastic properties are defined by the condition (Love 1944) that its strain energy volume density function has the form

$$2W = A(e^2_{xx} + e^2_{yy}) + C\, e^2_{zz} + 2F(e_{xx} + e_{yy})e_{zz} + 2(A-2N)\, e_{xx}\, e_{yy} + L(e^2_{yz} + e^2_{zx}) + N\, e^2_{xy} \qquad (1)$$

Below this layer is an isotropic homogeneous half space with density ρ_3 and elastic constants λ_3 and μ_3. Co-ordinate axes are taken as shown in figure (1) and the layers are labelled I, II, III as shown.



Restricting the motion to two dimensions (x, z), the strain energy volume density function (1) becomes

$$2W = Ae^2_{xx} + Ce^2_{zz} + 2Fe_{xx}\, e_{zz} + Le^2_{zx} \qquad (2)$$

Since W is a positive definite form, therefore

$$A > 0,\ C > 0,\ L > 0 \text{ and } AC > F^2 \qquad (3)$$

We shall further assume that

$$A > L \text{ and } C > L.$$

In medium I, which is a liquid layer of thickness h, the equation of motion in terms of displacement potential ϕ is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c_1^2}\ \frac{\partial^2\phi}{\partial t^2}, \qquad (4)$$

where $c_1 \left(= \sqrt{\frac{\lambda_1}{\rho_1}}\right)$ is the velocity of sound in the liquid. The displacement components u_1, w_1 and the pressure p are given by

$$u_1 = \frac{\partial\phi}{\partial x},$$

$$w_1 = \frac{\partial\phi}{\partial z}, \qquad (5)$$

$$p = -\rho_1 \frac{\partial^2\phi}{\partial t^2}.$$

For waves propagating in the direction ox, with wave length $2\pi/k$ and velocity c, we assume

$$\phi = \phi_1(z) \exp ik(x-ct) \tag{6}$$

Substituting ϕ from (6) in (4) we get

$$\frac{d^2\phi_1(z)}{dz^2} + k_o^2 \phi_1(z) = 0, \tag{7}$$

where k_o is real and given by

$$k_o^2 = k^2\left(\frac{c^2}{c_1^2} - 1\right) \tag{8}$$

Equation (7) gives

$$\phi_1(z) = B \cos(k_o z) + D \sin(k_o z) \tag{9}$$

where B and D are constants.

In medium II, which is a layer of homogeneous transversely isotropic elastic material of thickness H, components of stress can be derived from the strain energy volume density function by the formulae

$$T_{ij} = \frac{\partial w}{\partial e_{ij}}$$

$$T_{ii} = \frac{\partial w}{\partial e_{ii}} \text{ (no summation)}, \tag{10}$$

Thus we get from (2)

$$T_{xx} = \frac{\partial w}{\partial e_{xx}} = Ae_{xx} + Fe_{zz} = A\frac{\partial u_2}{\partial x} + F\frac{\partial w_2}{\partial z},$$

$$T_{zz} = \frac{\partial w}{\partial e_{zz}} = Ce_{zz} + Fe_{xx} = F\frac{\partial u_2}{\partial x} + C\frac{\partial w_2}{\partial z}, \tag{11}$$

$$T_{zx} = \frac{\partial w}{\partial e_{zx}} = Le_{zx} = L\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x}\right),$$

where u_2 and w_2 are the displacement components and e_{ij} etc. are replaced by their values in terms of displacement components.

The equation of motion where there are no body forces, are

$$\rho_2 \frac{\partial^2 u_2}{\partial t^2} = A\frac{\partial^2 u_2}{\partial x^2} + L\frac{\partial^2 u_2}{\partial z^2} + (F+L)\frac{\partial^2 w_2}{\partial z\,\partial x}, \tag{12}$$

$$\rho_2 \frac{\partial^2 w_2}{\partial t^2} = L\frac{\partial^2 w_2}{\partial x^2} + C\frac{\partial^2 w_2}{\partial z^2} + (F+L)\frac{\partial^2 u_2}{\partial z\,\partial x}. \tag{13}$$

If we substitute

$$A = C = \lambda + 2\mu, \quad F = \lambda, \quad L = \mu, \tag{14}$$

equations (12) and (13) reduced to the familiar equations for an isotropic body with Lame's constants λ and μ.

For the types of waves under consideration we seek the solution of the form

$$(u_2, w_2) = [U(z), W(z)] \exp[ik(x-ct)] \tag{15}$$

where k and c are as defined earlier.

These values of displacement components satisfy the equations (12) and (13) if

$$-\rho_2 k^2 c^2 U(z) = -AK^2 U(z) + LU''(z) + ik(F+L)W'(z), \qquad (16)$$

$$-\rho_2 k^2 c^2 W(z) = -LK^2 W(z) + CW''(z) + ik(F+L)U'(z), \qquad (17)$$

where primes denote differentiation with respect to z.

If we assume that U(z) and W(z) have the forms

$$U(z) = Ue^{mz},$$
$$W(z) = iWe^{mz}, \qquad (18)$$

the equations (16) and (17) reduce to

$$-\rho_2 k^2 c^2 U = -Ak^2 U + Lm^2 U - km(F+L)W, \qquad (19)$$

$$-i\rho_2 k^2 c^2 W = -iLK^2 W + iCm^2 W + ik(F+L)mU \qquad (20)$$

Equations (19) and (20) give us the ratio U:W, for which (15) is a solution of the equation of motion. In order that (19) and (20) give us a non-zero solution, we must have :

$$(Lm^2 - Ak^2 + \rho_2 k^2 c^2)(Cm^2 - Lk^2 + \rho_2 k^2 c^2) + k^2(F+L)^2 m^2 = 0 \qquad (21)$$

This equation is quadratic in m^2 and therefore gives two values of m^2 for which (15) is a solution. Let these be m_i^2 ($i = 1, 2$).

The ratio of displacement components U_i, W_i corresponding to $m = m_i$ is

$$\frac{U_i}{W_i} = \frac{km_i(F+L)}{Lm_i^2 - Ak^2 + \rho_2 k^2 c^2} = \frac{Cm_i^2 - Lk^2 + \rho_2 k^2 c^2}{k(F+L)m_i} = \frac{1}{\epsilon_i}\text{ (say)}. \qquad (22)$$

Thus we can write our solution as

$$U(z) = U_1 \sinh(m_1 z) + U_2 \cosh(m_1 z) + U_3 \sinh(m_2 z) + U_4 \cosh(m_2 z),$$
$$W(z) = i\epsilon U_1 \cosh(m_1 z) + i\epsilon_1 U_2 \sinh(m_1 z) + i\epsilon_2 U_3 \cosh(m_2 z) + i\epsilon_2 U_4 \sinh(m_2 z). \qquad (23)$$

Components of stress in this medium are, therefore, given by

$$T_{xx} = [iU_1(Ak + \epsilon_1 m_1)\sinh(m_1 z) + iU_2(Ak + \epsilon_1 m_1)\cosh(m_1 z) + iU_3(Ak + \epsilon_2 m_2)\sinh(m_2 z) + iU_4(Ak + \epsilon_2 m_2)\cosh(m_2 z)] \times \exp[ik(x-ct)],$$

$$T_{xz} = L[(m_1 - \epsilon_1 k)U_1 \cosh(m_1 z) + (m_1 - \epsilon_1 k)U_2 \sinh(m_1 z) + (m_2 - \epsilon_2 k)U_3 \cosh(m_2 z) + (m_2 - \epsilon_2 k)U_4 \sinh(m_2 z)]\exp[ik(x-ct)] \qquad (24)$$

$$T_{zz} = i[U_1(\epsilon_1 m_1 c + kF)\sinh(m_1 z) + U_2(\epsilon_1 m_1 c + kF)\cosh(m_1 z) + U_3(\epsilon_2 m_2 c + kF)\sinh(m_2 z) + U_4(\epsilon_2 m_2 c + kF)\cosh(m_2 z)]\exp[ik(x-ct)]$$

In the half space III, let the displacement components u_3 and w_3 along x and z-axes respectively be defined by the function ϕ and ψ such that :

$$u_3 = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z}$$
$$w_3 = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x} \qquad (25)$$

Then the equation of motion will be satisfied if ϕ and ψ satisfy the equations

$$\nabla^2 \phi = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2},$$

$$\nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \tag{26}$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \text{ and } \alpha^2 = \frac{\lambda_3 + 2\mu_3}{\rho_3}, \quad \beta^2 = \frac{\mu_3}{\rho_3} \tag{27}$$

For types of waves under consideration we assume

$$\phi = \phi_1(z) \exp\,[ik(x-ct)],$$

$$\psi = \psi_1(z) \exp\,[ik(x-ct)], \tag{28}$$

where k and c are as defined earlier.

Substituting the values of ϕ and ψ from equations (28) in (26) we get

$$\frac{d^2 \phi_1(z)}{dz^2} - k^2\left(1 - \frac{c^2}{\alpha^2}\right)\phi_1(z) = 0,$$

$$\frac{d^2 \psi_1(z)}{dz^2} - k^2\left(1 - \frac{c^2}{\beta^2}\right)\psi_1(z) = 0. \tag{29}$$

Since the displacement components are to tend to zero in this region as z tends to infinity, we therefore, take the solutions of equations (29) as :

$$\phi_1(z) = P\, e^{-rz}$$

$$\psi_1(z) = Q\, e^{-sz} \tag{30}$$

where

$$r^2 = k^2\left(1 - \frac{c^2}{\alpha^2}\right),$$

$$s^2 = k^2\left(1 - \frac{c^2}{\beta^2}\right). \tag{31}$$

Thus

$$u_3 = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} = -i\left(\frac{k}{r} P_1 e^{-rz} + \frac{s}{k} Q_1 e^{-sz}\right) \exp\, ik(x-ct) \tag{32}$$

$$w_3 = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x} = (P_1 e^{-rz} + Q_1 e^{-sz}) \exp\, ik(x-ct) \tag{33}$$

where $$P_1 = -Pr, \qquad Q_1 = -ikQ. \tag{34}$$

Components of stress in this medium are given by

$$T_{zz} = \lambda_3\left(\frac{\partial u_3}{\partial x} + \frac{\partial w_3}{\partial z}\right) + 2\mu_3 \frac{\partial w_3}{\partial z}$$

$$= \left[\left(\frac{\lambda_3 k^2}{r} - (\lambda_3 + 2\mu_3)\, r\right) P_1 e^{-rz} - 2\mu_3\, s Q_1 e^{-sz}\right] \exp\,[ik(x-ct)] \tag{35}$$

$$T_{zx} = \mu_3\left(\frac{\partial u_3}{\partial z} + \frac{\partial w_3}{\partial x}\right) = i\mu_3\left[2k\, P_1 e^{-rz} + \left(\frac{s^2}{k} + k\right) Q_1 e^{-sz}\right] \exp\,[ik(x-ct)] \tag{36}$$

Boundary Conditions

The boundary conditions are that the displacements and stresses are continuous at the interfaces and the pressure vanishes at the free surface of the liquid (tangential displacements are not required to be continuous, while the [illegible] which is [illegible]. This approximation is valid as long as the motions are small and there is assumed to be, at the liquid-solid interface, a viscous boundary layer whose thickness is small compared with the depth of the fluid. These assumptions hold in problems of transmission of surface waves in the ocean bed). We thus have eight conditions :

$$
\begin{array}{ll}
p = 0 & \text{on } z = -h \\
(T_{zz})_{II} = -p & \text{on } z = 0 \\
(T_{xz})_{II} = 0 & \text{on } z = 0 \\
W_2 = W_1 & \text{on } z = 0 \\
(T_{zz})_{II} = (T_{zz})_{III} & \text{on } z = h' \\
(T_{xz})_{II} = (T_{xz})_{III} & \text{on } z = h' \\
w_2 = w_3 & \text{on } z = h' \\
u_2 = u_3 & \text{on } z = h'
\end{array}
\qquad (37)
$$

Making use of the equations (5), (9), (11), (15), (23), (24), (32), (33), (34) and (35), we get from the equation (37) :

$$B \cos (k_0 h) - D \sin (k_0 h) = 0, \qquad R_1 U_2 + R_2 U_4 + [illegible] B = 0,$$

$$S_1 U_1 + S_2 U_3 = 0, \quad [illegible] = 0,$$

$$R_1 U_1 \sinh (m_1 h') + R_1 U_2 \cosh (m_1 h') + R_2 U_3 \sinh (m_2 h') + R_2 U_4 \cosh (m_2 h')$$

$$+ i \left[\frac{\lambda_3 k^2}{r} - (\lambda_3 + 2\mu_3)\, r \right] P_1 e^{-rh'} - 2i\, \mu_3\, S Q_1 e^{-sh'} = 0, \qquad (38)$$

$$S_1 U_1 \cosh (m_1 h') + S_1 U_2 \sinh (m_1 h') + S_2 U_3 \cosh (m_2 h') + S_2 U_4 \sinh (m_2 h')$$

$$-2\, i\, \mu_3\, k\, P_1 e^{-rh'} - i\, \mu_3 \left(\frac{s^2}{k} + k \right) Q_1 e^{-sh'} = 0$$

$$\epsilon_1 U_1 \cosh (m_1 h') + \epsilon_1 U_2 \sinh (m_1 h') + \epsilon_2 U_3 \cosh (m_2 h') + \epsilon_2 U_4 \sinh (m_2 h')$$

$$+ i P_1 e^{-rh'} + i Q_1 e^{-sh'} = 0$$

$$U_1 \sinh (m_1 h') + U_2 \cosh (m_1 h') + U_3 \sinh (m_2 h') + U_4 \cosh (m_2 h')$$

$$+ i \frac{k}{r} P_1 e^{-rh'} + i \frac{S}{K} Q_1 e^{-sh'} = 0$$

where

$$R_1 = kF + C\, \epsilon_1 m_1 \qquad R_2 = kF + C\, \epsilon_2 m_2,$$

$$S_1 = L\,(m_1 - k\, \epsilon_1) \qquad S_2 = L\,(m_2 - k\, \epsilon_2) \qquad (39)$$

The elimination of the eight constants U_1, U_2, U_3, U_4, P_1, Q_1, B and D among the equations (38) gives the frequency equation as :

This is the equation relating the phase velocity c, to the wavelength $\frac{2\pi}{k}$. The wavelength is a multivalued function of the phase velocity, each value corresponding to different mode of propagation. Given any value of k, c can be determined from (40) and subsequently the ratios of the constants can be determined from the equations (38) and hence the amplitude of motion may be determined.

$$
\begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cos(k_o h) & \sin(k_o h) \\
0 & R_1 & 0 & R_2 & 0 & 0 & \rho_1 k^2 c^2 & 0 \\
S_1 & 0 & S_2 & 0 & 0 & 0 & 0 & 0 \\
\epsilon_1 & 0 & \epsilon_2 & 0 & 0 & 0 & 0 & k_o \\
R_1 \sinh(m_1 h') & R_1 \cosh(m_1 h') & R_2 \sinh(m_2 h') & R_2 \cosh(m_2 h') & \frac{\lambda_s k^2}{r} - (\lambda_s + 2\mu_s) r & -2\mu_s s & 0 & 0 \\
S_1 \cosh(m_1 h') & S_1 \sinh(m_1 h') & S_2 \cosh(m_2 h') & S_2 \sinh(m_2 h') & -2\mu_s k & -\mu_s\left(\frac{s^2}{k} + k\right) & 0 & 0 \\
\epsilon_1 \cosh(m_1 h') & \epsilon_1 \sinh(m_1 h') & \epsilon_2 \cosh(m_2 h') & \epsilon_2 \sinh(m_2 h') & 1 & 1 & 0 & 0 \\
\sinh(m_1 h') & \cosh(m_1 h') & \sinh(m_2 h') & \cosh(m_2 h') & \frac{k}{r} & \frac{s}{k} & 0 & 0
\end{vmatrix} = 0 \qquad (4[illegible])
$$

$$
\begin{vmatrix}
0 & R_1 & 0 & R_2 & 0 & 0 \\
S_1 & 0 & S_2 & 0 & 0 & 0 \\
R_1 \sinh(m_1 h') & R_1 \cosh(m_1 h') & R_2 \sinh(m_2 h') & R_2 \cosh(m_2 h') & \frac{\lambda_s k^2}{r} - (\lambda_s + 2\mu_s) r & -2\mu_s s \\
S_1 \cosh(m_1 h') & S_1 \sinh(m_1 h') & S_2 \cosh(m_2 h') & S_2 \sinh(m_2 h') & -2\mu_s k & -\mu_s\left(\frac{s^2}{k} + k\right) \\
\epsilon_1 \cosh(m_1 h') & \epsilon_1 \sinh(m_1 h') & \epsilon_2 \cosh(m_2 h') & \epsilon_2 \sinh(m_2 h') & 1 & 1 \\
\sinh(m_1 h') & \cosh(m_1 h') & \sinh(m_2 h') & \cosh(m_2 h') & \frac{k}{r} & \frac{s}{k}
\end{vmatrix} = 0 \qquad (4[illegible])
$$

The group velocity can be derived from the frequency equation (40) by the formula

$$U = c + k\,\frac{dc}{dk}\,. \tag{41}$$

Special Cases

(i) If we reduce the anisotropic layer to zero substituting $h' = o$ in the equation (40), we obtain the frequency equation as :

$$\tan (k_0 h) = \frac{\rho_2}{\rho_1}\left(\frac{\beta}{w}\right)^4 \frac{k_0}{r}\left[\,4rs\,k^2-(s^2+k^2)^2\right], \tag{42}$$

which is the dispersion equation of liquid overlying a semi-infinite homogeneous isotropic elastic solid (Tolstoy (1954), Ewing et al. , 1957, p. 161)

As $kh \to o$ in (42) both U and c tend to the same value c_R. c_R is the solution of

$$4\left(1-\frac{c^2}{\alpha^2}\right)\left(1-\frac{c^2}{\beta^2}\right)-\left(2-\frac{c^2}{\beta^2}\right)^2 = 0, \tag{43}$$

and is the velocity of Rayleigh waves in the solid medium with a free upper surface. The result is understandable physically since $kh \to o$ means that the waves have wavelengths large compared with the depth of the liquid layer and thus the effect of layer is negligible.

As $kh \to \infty$ in (42), that is the wave length becomes small compared with the depth of the liquid layer, the phase and group velocities tend to the common value c_s, the velocity of the Stoneley waves at the interface of solid and liquid, when the liquid is semi-infinite in extent. c_s is the solution of

$$\sqrt{\frac{c^2}{\alpha^2}-1} = \frac{\rho_2}{\rho_1}\,\frac{\beta^4}{c^4}\sqrt{\frac{c^2}{c_1^2}-1}\left[\left(2-\frac{c^2}{\beta^2}\right)^2-4\left(1-\frac{c^2}{\alpha^2}\right)^{\frac{1}{2}}\left(1-\frac{c^2}{\beta^2}\right)^{\frac{1}{2}}\right]. \tag{44}$$

(ii) If we reduce the liquid layer to zero by substituting $h = o$ in the frequency equation (40) we obtain Eq. (45).

Equation (45) gives us the frequency equation for Rayleigh type waves in a transversely isotropic layer lying over an isotropic half-space. As far as the author is aware, this equation has not been given before.

Numerical Calculations

For the purpose of calculations the variables were changed to

$$y = kh, \tag{46}$$

$$x = c/c_1, \tag{47}$$

and the frequency equation (40) was written as :

$$|\,a_{ij}\,| = 0 \tag{48}$$

where

$$a_{11} = a_{12} = a_{13} = a_{14} = a_{15} = a_{16} = a_{21} = a_{23} = a_{25} = a_{26} = a_{28} = a_{32} =$$
$$a_{34} = a_{36} = a_{37} = a_{38} = a_{42} = a_{44} = a_{45} = a_{46} = a_{47} = a_{57} = a_{58} = a_{67} =$$
$$a_{68} = a_{77} = a_{78} = a_{87} = a_{88} = 0,$$

$$a_{17} = \cos\left(y\sqrt{x^2-1}\right), \qquad a_{18} = \sin\left(y\sqrt{x^2-1}\right),$$

$$a_{23} = \frac{\frac{F}{C} + \frac{-G + \sqrt{G^2 - 4LCH}}{2C} - A + \rho_2 x^2 c_1^2}{F + L},$$

$$a_{24} = \frac{\frac{F}{C} + \frac{-G - \sqrt{G^2 - 4LCH}}{2C} - A + \rho_2 x^2 c_1^2}{F + L}$$

$$a_{27} = \frac{F}{C} \quad \frac{\rho_1 c_1^2}{C} x^2,$$

$$a_{31} = \frac{(F/2LC)\left(-G + \sqrt{G^2 - 4LCH}\right) + A - \rho_2 c_1^2 x^2}{(F+L)\left[-G + (G^2 - 4LCH)^{1/2}\right]^{1/2}},$$

$$a_{33} = \frac{(F/\sqrt{2LC})\left(-G - \sqrt{G^2 - 4LCH}\right) + A - \rho_2 c_1^2 x^2}{(F+L)\left[-G - (G^2 - 4LCH)^{1/2}\right]^{1/2}},$$

$$a_{41} = \frac{-G + \sqrt{G^2 - 4LCH} - 2C\left(A - \rho_2 c_1^2 x^2\right)}{\left[-G + (G^2 - 4LCH)^{1/2}\right]^{1/2}} \; \frac{L}{F+L},$$

$$a_{43} = \frac{-G \quad \sqrt{G^2 - 4LCH} - 2C\left(A - \rho_2 c_1^2 x^2\right)}{\left[-G - (G^2 - 4LCH)^{1/2}\right]^{1/2}} \; \frac{L}{F+L},$$

$a_{48} = \sqrt{x^2 - 1},$ $\qquad a_{51} = a_{23} \sinh(\theta_1),$

$a_{52} = a_{23} \cosh(\theta_1),$ $\qquad a_{53} = a_{24} \sinh(\theta_2),$

$a_{54} = a_{24} \cosh(\theta_2),$ $\qquad a_{55} = \frac{\lambda_2}{c}\left(1 - \frac{c_1^2}{\alpha^2} x^2\right)^{1/2} - \frac{\lambda_2 + 2\mu_2}{c}\left(1 - \frac{c_1^2}{\alpha^2} x^2\right)^{1/2},$

$a_{56} = -\frac{2\mu_2}{c}\left(1 - \frac{c_1^2}{\beta^2} x^2\right)^{1/2},$ $\qquad a_{61} = a_{31} \cosh(\theta_1),$

$a_{62} = a_{31} \sinh(\theta_1),$ $\qquad a_{63} = a_{33} \cosh(\theta_2),$

$a_{64} = a_{33} \sinh(\theta_2),$ $\qquad a_{65} = -\frac{2\mu_2}{L}$

$a_{66} = -\frac{\mu_2}{L}\left(2 - \frac{c_1^2}{\beta^2} x^2\right)$ $\qquad a_{71} = a_{41} \cosh(\theta_1),$

$a_{72} = a_{41} \sinh(\theta_1),$ $\qquad a_{73} = a_{43} \cosh(\theta_2),$

$a_{74} = a_{43} \sinh(\theta_2),$ $\qquad a_{75} = a_{76} = 1,$

$a_{81} = \sinh(\theta_1),$ $\qquad a_{82} = \cosh(\theta_1),$

$a_{83} = \sinh(\theta_2),$ $\qquad a_{84} = \cosh(\theta_2),$

$a_{85} = \left(1 - \frac{c_1^2}{\alpha^2} x^2\right)^{1/2}$ $\qquad a_{86} = \left(1 - \frac{c_1^2}{\beta^2} x^2\right)^{1/2}$

$$G = (F + L)^2 - (A - \rho_2 c_1^2 x^2) C - L (L - \rho_2 c_1^2 x^2),$$

$$H = (A - \rho_2 c_1^2 x^2)(L - \rho_2 c_1^2 x^2),$$

$$\theta_1 = \frac{h'}{h} y \left[\frac{-G + \sqrt{G^2 - 4LCH}}{2LC}\right]^{1/2}$$

$$\theta_2 = \frac{h'}{h} y \left[\frac{-G - \sqrt{G^2 - 4LCH}}{2LC}\right]^{1/2}$$

Numerical calculations of y in terms of x were made at the Cambridge University Mathematical Laboratory on TITAN. The group velocity was also calculated from the formula

$$U = c + k\frac{dc}{dk}, \tag{49}$$

that is $\frac{U}{c_1} = x + y\frac{dx}{dy}$. (50)

Elastic constants A, C, F, L should be those for the sediments of the ocean bed. But no observational results for these are available. For want of better material, I have used the elastic constants for beryl, but when more appropriate values become available they should be inserted in place of those used here. These were therefore used and phase velocities and group velocities were calculated for different wave numbers.

In all calculations (Love 1944)

$$A = 26.94 \quad C = 23.63 \quad F = 6.61 \quad L = 6.53$$

measured in 10^{11} dynes/cm^2.

The thickness of the liquid layer and anisotropic layer in these calculations are taken to be 5 km. and 4 km. respectively.

The phase velocities and the group velocities for different values of the wave number k are given in Table I and exhibited in Fig. 2. In this figure dispersion curve for only the fundamental mode is given. My original intention was to calculate the dispersion curves for several modes, but the computation of the first mode alone occupied one hour of machine time and so it was decided not to calculate the dispersion curves for the other modes.

Fig (2) shows the curves of the phase and group velocities as they vary with wave number in the fundamental mode. These appear to be similar in every respect to the curves drawn by Tolstoy (1954) for the completely isotropic case and Abubaker & Hudson (1961) for a liquid layer overlying a transversely isotropic half space.

An infinite number of modes higher than the fundamental exist for $L/\rho_2 > c^2$, $c > c_1$.

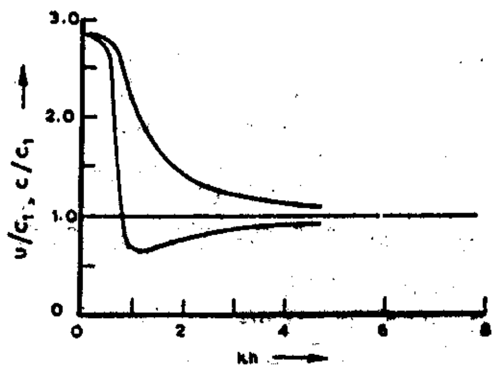


Fig. 2 The variation of phase and group velocities of the fundamental mode with wave number

It is to be expected that they will also be similar to those drawn by Tolstoy (1954) and Abubaker & Hudson (1961).

As the wave length becomes small compared with the depth of the liquid layer, the phase and group velocities of every mode tend to the common value c_1, the velocity of compressional waves in the liquid. On the other hand, it is clear from the Fig. (2) that as kh → o both U and c tend to the value c_R which is the velocity of the Rayleigh-type waves in the transversely isotropic layer overlying an isotropic half space.

Table 1

Variation of phase and group velocity with wave number for the fundamental mode

kh	C/C_1	U/C_1	kh	C/C_1	U/C_1	kh	C/C_1	U/C_1
0.10	2.81	2.80	1.7	1.53	0.70	3.3	1.17	0.87
0.20	2.80	2.78	1.8	1.48	0.72	3.4	1.17	0.88
0.30	2.79	2.76	1.9	1.44	0.73	3.5	1.16	0.88
0 40	2.78	2.72	2.0	1.41	0.75	3.6	1.15	0.89
0.50	2.76	2.62	2.1	1.37	0.76	3.7	1.14	0.89
0.60	2.72	2.35	2.2	1.35	0.78	3.8	1.14	0.89
0.70	2.63	1.74	2.3	1.32	0.79	3.9	1.13	0.90
0.80	2 47	1.09	2.4	1.30	0.80	4.0	1.12	0.90
0.90	2.29	0.77	2.5	1.28	0.81	4.1	1.12	0.91
1.00	2.14	0.66	2.6	1.26	0.82	4.2	1.11	0.91
1.10	2.00	0.63	2.7	1.25	0.83	4.3	1.11	0.91
1.20	1.89	0.63	2.8	1.23	0.84	4.4	1.10	0.92
1.30	1.79	0.64	2 9	1.22	0.84	4.5	1.10	0.92
1.40	1.71	0.65	3.0	1.21	0.85	4.6	1.10	0.92
1.50	1.64	0.67	3.1	1.19	0.86	4.7	1.09	0.92
1.60	1.58	0.68	3.2	1.18	0.86	4.8	1.09	0.93

Conclusions

The above calculations show that a layer of transversely isotropic material underneath a uniform liquid layer and overlying an isotropic half-space has little effect on the shape of the dispersion curves of surface waves of Rayleigh-type if we use the elastic constants of beryl crystal to characterise the transversely isotropic layer. This conclusion might not hold if constants appropriate to actual rocks below the sea bottom were inserted. Experimental determination of these constants is much needed. If they were available, calculations like those here could give dispersion curves for comparasion with observations.

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