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VIBRATION OF SHALLOW SPHERICAL SHELLS RECTANGULAR IN PLAN

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INTRODUCTION

Vibration problems in engineering have got tremendous importance now-a-days. Shells belong to the most useful class of structures and as such, their dynamic behaviour is definitely worth investigating.

The basic equations of equilibrium or of motion of shallow shells[8] consist of two coupled fourth degree partial differential equations in terms of a stress function and the normal deflection function. These equations were used by Nowacki[5] to study the free vibration of shallow spherical shells. The Marguerre[4] equations as well can be used suitably for shells rectangular in plan. These equations of motion of a shell element consist of three equations in terms of three displacement functions u, v, w. As such, in solving the system it will be required to use all the three equations simultaneously. It has, however, been proved by the author[2], without any simplifying assumption, that the basic equations of equilibrium (and hence of motion) for a shallow spherical shell can be simplified to the system

$$\nabla u = \lambda \frac{\partial w}{\partial x},$$

$$\nabla v = \lambda \frac{\partial w}{\partial y},$$

$$\nabla\nabla w + \mu^2 w = -\frac{\gamma h}{gD}\frac{\partial^2 w}{\partial t^2} \quad (1)$$

where, ∇=L aplacian operator,
γ=density of the material
h=thickness of the shell
g=acceleration due to gravity
D=flexural rigidity of shell element,
γ=Poisson's ratio,
t=time, E = Young's modulus,
R=radius of curvature of the shell
and, $\lambda=(1+\nu)/R$, $\mu^2=Eh/DR^2$

Now, it is obvious, that the third equation of the system (1) is independent of the other two, and it represents the equation of motion of a thin plate on Winkler type elastic foundation if the spring constant is assumed to be equal to Eh/R^2. For investigating the transverse vibration of the shell equation of the system (1) is adequate by itself.

NATURAL FREQUENCIES OF VIBRATION

The natural frequencies of vibration of the shell under various combinations of boundary conditions are computed below. The vibration being harmonic, displacement function w can be assumed as follows,

$$w(x, y, t) = W(x, y)\, e^{ipt} \quad (2)$$

where, $W(x, y)$=unknown function,
and, p=frequency of vibration,
$i=\sqrt{-1}$.

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Inserting the equation 2 into the last equation of the system (1) :

$$\nabla\nabla W-\beta^2 W=0 \tag{3}$$

where,

$$\beta^2=\frac{\gamma h}{gD}p^2-\mu^2$$

CASE 1 : *All edges simply supported*

Assume

$$W(x, y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{4}$$

where a, b are rectangular plan dimensions.

Inserting expression (4) into the equation 3, the following expression for frequency of vibration is obtained,

$$p_{mn}^2=\frac{g}{\gamma h}\left\{D\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2+\frac{Eh}{R^2}\right\} \tag{5}$$

which will give the spectre of natural frequencies of the shell. The lowest value would, of course, be given for $m=n=1$.

As a particular case, the spectre of frequencies of thin rectangular plate can be obtained by letting $R \longrightarrow \infty$. In this case,

$$p_{mn}=\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)\sqrt{\frac{gD}{\gamma h}} \tag{6}$$

which conicides with the well known result.[7]

CASE 2 : *Two opposite edges are simply supported*

Assume,

$$W(x, y)=\sum_{m=0}^{\infty} Y_m(y)\sin\frac{m\pi x}{a} \tag{7}$$

where Y_m is unknown function to be determined using appropriate boundary conditions along edges $y=0$, $y=b$, whereas the edges $x=0$, $x=a$ are taken to be simply supported.

The expression (7) when inserted into the equation 3 will produce the differential equation,

$$Y_m^{IV}-2\left(\frac{m\pi}{a}\right)^2 Y_m^{II}+\left\{\left(\frac{m\pi}{a}\right)^4-\beta^2\right\}Y_m=0 \tag{8}$$

Assuming that, $m^2\pi^2/a^2>\beta$,

$$Y_m(y)=A_m\, sh(\lambda_m y)+B_m\, ch(\lambda_m y)+C_m\, sh(\mu_m y)+D_m\, ch(\mu_m y) \tag{9}$$

where,

$$\lambda_m=\sqrt{\frac{m^2\pi^2}{a^2}-\beta}$$

$$\mu_m=\sqrt{\frac{m^2\pi^2}{a^2}+\beta}$$

and, A_m, B_m, C_m, D_m—arbitrary constants.

If, in particular, the edge $y=0$ is simply supported and the edge $y=b$ is clamped, the boundary conditions will be as follows,

$$Y_m(0)=Y_m''(0)=0$$

and

$$Y_m(b)=Y_m'(b)=0 \tag{9a}$$

These conditions when used in the expression (9) will give four homogeneous equations containing the four unknown constants of integration. The determinant of these coefficients made equal to zero gives the equation for determining frequencies of vibration. In this particular case, the frequency will be given by:

$$\mu_m \tanh(\lambda_m b) = \lambda_m \tanh(\mu_m b) \tag{10}$$

If however, $m^2\pi^2/a^2 < \beta$, the solution of equation (8) will become

$$Y_m(y) = A_m \sin(\bar{\lambda}_m y) + B_m \cos(\bar{\lambda}_m y) + $$
$$+ C_m\, sh(\mu_m y) + D_m\, ch(\mu_m y)$$

where,

$$\bar{\lambda}_m = \sqrt{-m^2\pi^2/a^2 + \beta}$$

The frequency of free vibration of the shell satisfying the boundary conditions (9a) in this case will be determined by:

$$\mu_m \tan(\bar{\lambda}_m b) = \bar{\lambda}_m \tanh(\mu_m b) \tag{10a}$$

In the case when $m^2\pi^2/a^2 = \beta$,

degeneracy occurs, since the frequency of vibration is directly obtainable regardless of the shell boundary conditions which contradicts the physical concepts.

CASE 3. *All edges clamped*

The approximate method namely the variational method due to Vlasov–Kantorovich[3] shall be used here.

Assume,

$$W(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \left\{1 - \frac{\cos(\xi_m x)}{\cos(m\pi)}\right\}\left\{1 - \frac{\cos(\xi_n y)}{\cos(n\pi)}\right\} \tag{11}$$

where

$$\xi_m = 2m\pi/a, \quad \xi_n = 2n\pi/b.$$

Substituting the expression (11) into the equation (3), multiplying by

$$\left\{1 - \frac{\cos(\xi_m x)}{\cos(m\pi)}\right\}\left\{1 - \frac{\cos(\xi_n y)}{\cos(n\pi)}\right\}$$

and then integrating to the whole plane-area of the shell, we finally get the expression for the spectre of natural frequencies of vibrations:

$$p_{mn}^2 = \frac{g}{\gamma h}\left\{\frac{D}{9}\left(3\xi_m^4 + 2\xi_m^2\xi_n^2 + 3\xi_n^4\right) + \frac{Eh}{R^2}\right\} \tag{12}$$

Making $R \to \infty$, the frequencies for plates, clamped along all sides will be obtained.

FORCED VIBRATIONS

In the case of forced vibration, the last equation of the system (3) would become:

$$\nabla\nabla w + \mu^2 w + \frac{\gamma h}{gD}\frac{\partial^2 w}{\partial t^2} = \frac{1}{D} q(x,y,t) \tag{13}$$

where,

$q(x,y,t)$ is the external arbitrary load depending on time.

The solution of equation (13) can be written as

$$w(x,y,t) = \Sigma\, W_i(x,y)\phi_i(t) \tag{14}$$

where, $W_i(x,y)$ should satisfy all the geometrical as well as statical boundary conditions and, $\phi_i(t)$ should satisfy initial conditions.

Substituting expression (14) into equation (13), and using the relation (3) following equation is obtained

$$\sum \frac{\gamma h}{gD} W_i \left\{\ddot{\phi}_i + p_i^2 \phi i\right\} = \frac{1}{D} q(x,y,t) \tag{15}$$

The equation 15 is now multiplied by W_i and the product so obtained is integrated over the surface of the shell. This gives:

$$\ddot{\phi}_i+p_i^2\,\phi_i=Q_i(t) \tag{16}$$

where, $$Q_i(t)=\frac{g}{\xi\gamma h A_{mn}}\iint q(x,y,t)W_i(x,y)dx\ dy, \tag{16a}$$

with, $$\xi=\iint W_i^2(x,y)\,dx\ dy.$$

The solution of equation (16) is given by:

$$\phi_i(t)=A_i\cos p_i t+B_i\sin p_i t+\frac{1}{p_i}\int_0^t Q_i(\tau)\sin p_i(t-\tau)d\tau \tag{17}$$

where A_i and B_i are constants of integration which are to be determined from the initial conditions. Supposing that these conditions are:

$$w(x,y,0)=\theta(x,y),\ \dot{w}(x,y,0)=\psi(x,y)$$

the use of relation (14) will give:

$$\left.\begin{aligned}\theta(x,y)&=\Sigma W_i(x,y)\phi_i(0)\\ \psi(x,y)&=\Sigma W_i(x,y)\dot{\phi}_i(0)\end{aligned}\right\} \tag{18}$$

Multiplying these by $W_i(x,y)$ both sides and integrating over the whole surface of the shell,

$$\xi\phi_i(0)=\iint\theta(x,y)\ W_i(x,y)dx\ dy \tag{19}$$

$$\xi\dot{\phi}_i(0)=\iint\psi(x,y)W_i(x,y)dx\ dy$$

Using the expression (17) in (19), given,

$$A_i=\phi_i(0),\ B_i=\frac{1}{p_i}\ \dot{\phi}_i(0),$$

and therefore :

$$A_i=\frac{1}{\xi}\iint\theta(x,y)\ W_i(x,y)\ dx\,dy$$

$$B_i=\frac{1}{p_i\xi}\iint\psi(x,y)\ W_i(x,y)\ dx\ dy \tag{19a}$$

PARTICULAR CASES

CASE 1. *Simply supported shell with sinusoidal load*

Here, $$q(x,y,t)=q(x,y)\sin\Omega t.$$

Assume:

$$W(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

so that $$\xi=\int_0^a\int_0^b\sin^2\frac{m\pi x}{a}\ \sin^2\frac{n\pi y}{b}\ dx\,dy=\frac{ab}{4}$$

and $$Q_{mn}(t)=\frac{g\sin\Omega t}{\xi\gamma hA_{mn}}\int_0^a\int_0^b q(x,y)\ \sin\frac{m\pi x}{a}\ \sin\frac{n\pi y}{b}\ dx\,dy$$

$$=\frac{g\sin\Omega t}{\gamma h}\cdot\frac{B_{mn}}{A_{mn}},$$

where, $$B_{mn}=\frac{16q}{\pi^2 mn} \quad \text{when} \quad q\,(x,y)=\text{constant}$$

and $$B_{mn}=\frac{4P}{ab}\sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\delta}{b}$$

when there is a point load P sin Ωt at (α, δ).

Moreover, $$\phi_{mn}(t)=\frac{g\,B_{mn}}{\gamma h A_{mn}}\cdot\frac{1}{p_{mn}}\int_0^t \sin\Omega t \sin p_{mn}(t-\tau)\,d\tau. \quad \ldots(20)$$

From equation (20) it follows,

$$\phi_{mn}(t)=\frac{g}{\gamma h}\cdot\frac{B_{mn}}{A_{mn}}\cdot\frac{\sin\Omega t}{p_{mn}^2-\Omega^2} \quad \ldots(21)$$

so that the amplitude of vibration will be given by:

$$W(x,y)=\frac{g}{\gamma h}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B_{mn}}{p_{mn}^2-\Omega^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad \ldots(22)$$

CASE 2. *Clamped shell with sinusoidal load*

Here again,

$$q\,(x,y,t)=q\,(x,y)\sin\Omega t$$

Assume:

$$W(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}\left\{1-\frac{\cos(\xi_m x)}{\cos(m\pi)}\right\}\left\{1-\frac{\cos(\xi_n y)}{\cos(n\pi)}\right\}$$

so that, $$\xi=\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\left\{1-\frac{\cos(\xi_m x)}{\cos(m\pi)}\right\}^2\left\{1-\frac{\cos(\xi_n y)}{\cos(n\pi)}\right\}^2 dx\,dy$$
$$=9/4\,ab$$

and, $$Q_{mn}(t)=\frac{g\sin\Omega t}{\xi\gamma h A_{mn}}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}q\,(x,y)\left\{1-\frac{\cos(\xi_m x)}{\cos(m\pi)}\right\}\left\{1-\frac{\cos(\xi_n y)}{\cos(n\pi)}\right\}dx\,dy$$
$$=\frac{g}{\gamma h}\cdot\frac{B_{mn}}{A_{mn}}\sin\Omega t \quad \ldots(23)$$

where,

$$B_{mn}=\frac{4}{9}q \quad \text{for} \quad q\,(x,y)=\text{constant}$$

$$B_{mn}=\frac{4P}{9ab}\left\{1-\frac{\cos(\xi_m\alpha)}{\xi_m\cos(m\pi)}\right\}\left\{1-\frac{\cos(\xi_n\delta)}{\xi_n\cos(n\pi)}\right\} \quad \ldots(24)$$

for a point load P sin Ωt at (α, δ).

Proceeding as before, the amplitude of vibration is obtained as:

$$W(x,y)=\frac{g}{\gamma h}\sum\sum\frac{B_{mn}}{p_{mn}^2-\Omega^2}\left\{1-\frac{\cos(\xi_m x)}{\cos(m\pi)}\right\}\left\{1-\frac{\cos(\xi_n y)}{\cos(n\pi)}\right\} \quad \ldots(25)$$

CASE 3. *Simply supported shell with a suddenly applied patch load*

Let a suddenly-applied uniformly distributed load in the form of a rectangle (Fig. 3) act on a simply supported shell.
Thus,

$$q\,(x,y,t)=q_0\,H(t), \quad \text{for} \quad \begin{cases}\bar{u}-\eta/2<x<\bar{u}+\eta/2\\ \bar{v}-\varepsilon/2<y<\bar{v}+\varepsilon/2\end{cases}$$

and,

$$q(x, y, t)=0, \quad \text{for} \quad \begin{cases} 0 < x < \bar{u}-\eta/2, \ \bar{u}+\eta/2 < x < a \\ 0 < y < \bar{v}-\varepsilon/2, \ \bar{v}+\varepsilon/2 < y < b \end{cases} \qquad \ldots(26)$$

where H (t) denotes the unit step function and q_0=constant. Substituting equation (26) into equation (16 a), and using the function W (x, y) for the simply-supported case,

$$Q_{mn}(t)=\frac{g}{\gamma h}\cdot\frac{16q_0}{ab}\cdot\frac{H(t)}{A_{mn}\pi^2 mn} T_1(\bar{u})\, T_2(\bar{v})\, T_3(\eta)\, T_4(\varepsilon) \qquad \ldots(27)$$

where,

$$T_1(\bar{u})=\sin\frac{m\pi\bar{u}}{a}, \quad T_2(\bar{v})=\sin\frac{n\pi\bar{v}}{b}$$

$$T_3(\eta)=\sin\frac{m\pi\eta}{2a}, \quad T_4(\varepsilon)=\sin\frac{n\pi\varepsilon}{2b}$$

Substituting equation (27) into equation (17),

$$\phi_{mn}(t)=\frac{g}{\gamma h}\cdot\frac{16q_0}{ab}\,\frac{(1-\cos p_{mn}t)}{A_{mn}\pi^2 mn p^2_{mn}} T_1(\bar{u})\, T_2(\bar{v})\, T_3(\eta)\, T_4(\varepsilon) \qquad \ldots(28)$$

Thus, the final solution would be:

$$W(x, y, t)=\sum\sum G_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \ldots(29)$$

where,

$$G_{mn}(t)=A_{mn}\,\phi_{mn}(t) \qquad \ldots(30)$$

The bending moment at any section will be given by:

$$M_x=D\sum\sum\left(\frac{m^2\pi^2}{a^2}+\nu\frac{n^2\pi^2}{b^2}\right) G_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$M_y=D\sum\sum\left(\frac{n^2\pi^2}{b^2}+\nu\frac{m^2\pi^2}{a^2}\right) G_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad \ldots(31)$$

CASE 4. *Clamped shell with a suddenly applied patch load*

Proceeding as before,

$$Q_{mn}(t)=\frac{g}{\xi\gamma h}\frac{q_0 H(t)}{A_{mn}}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\left\{1-\frac{\cos(\xi_m x)}{\cos(m\pi)}\right\}\left\{1-\frac{\cos(\xi_n y)}{\cos(n\pi)}\right\}dx\,dy$$

$$=\frac{g}{\xi\gamma h}\frac{q_0 H(t)}{A_{mn}}\left\{\eta-T_1(\bar{u})\,T_2(\eta)\right\}\left\{\varepsilon-T_3(\bar{v})\,T_4(\varepsilon)\right\}$$

where,

$$T_1(\bar{u})=\frac{2\sin(\xi_m\bar{u})}{\xi_m\cos(m\pi)}, \quad T_2(\eta)=\sin(\xi_m\eta/2)$$

$$T_3(\bar{v})=\frac{2\sin(\xi_n\bar{v})}{\xi_n\cos(n\pi)}, \quad T_4(\varepsilon)=\sin(\xi_n\varepsilon/2)$$

Equation (17) will be give,

$$\phi_{mn}(t)=\frac{g}{\gamma h}\cdot\frac{4q_0}{9ab}\,\frac{(1-\cos p_{mn}t)}{A_{mn}p^2_{mn}} F_{mn}(\bar{u}, \bar{v}, \eta, \varepsilon)$$

where, $F_{mn}=\{\eta-T_1T_2\}\,\{\varepsilon-T_3T_4\}$

Thus, the final solution would be:

$$W(x, y, t)=\sum\sum G_{mn}(t)\left\{1-\frac{\cos(\xi_m x)}{\cos(m\pi)}\right\}\left\{1-\frac{\cos(\xi_n y)}{\cos(n\pi)}\right\} \qquad \ldots(32)$$

where, $G_{mn}(t)=A_{mn}\,\phi_{mn}(t)$.

NUMERICAL RESULTS

Two shallow spherical shells with the given data

$$\nu = 0.12, \; E = 2\times10^9 \text{ kg/m}^2,$$

$$g = 9.81 \text{ m/sec}^2, \; \gamma = 2400 \text{ kg/m}^3, \; h = 0.1 \text{ m}$$

have been considered. Their plan dimensions are 10×10 m and 10m×6m. Figures 1 and 2 show the change in natural frequency (fundamental) due to the change in radius of curvatures of the shell for simply-supported as well as clamped boundaries. It is obvious from the figures that effect of clamping condition is more marked when radius is higher.

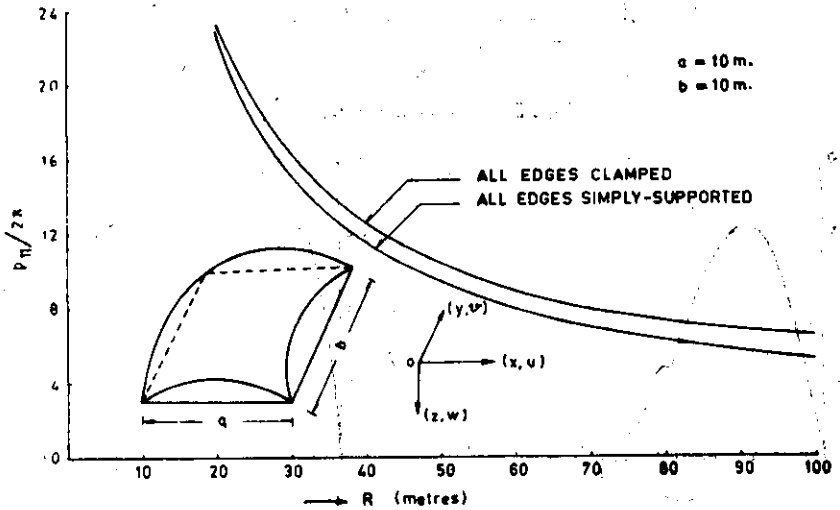

Fig. 1—Frequency Versus Radius

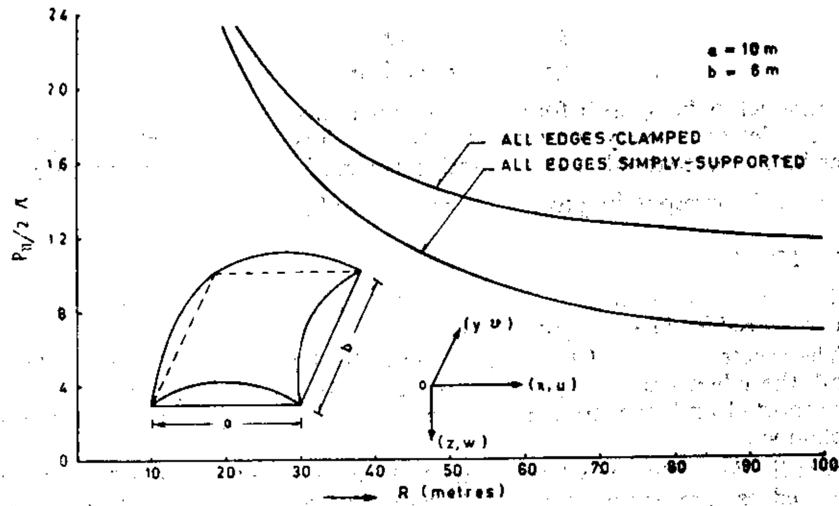

Fig. 2—Frequency Versus Radius

Figures 4 and 5 show the time history of centre-deflection of a 10 m×6m shell under a suddenly applied patch load with

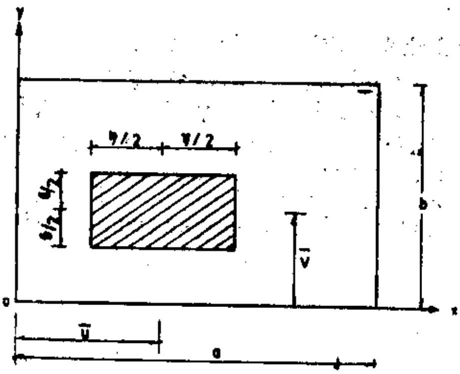

Fig. 3—Rectangular Patch Load

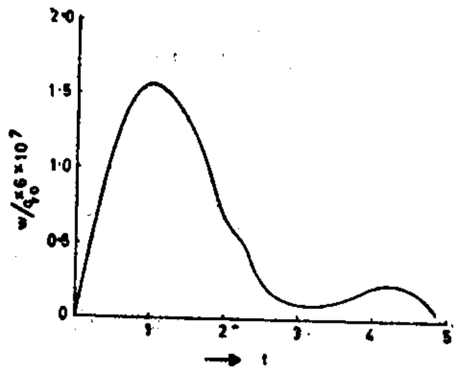

Fig. 4—Centre Deflection vs Time (Simply Supported)

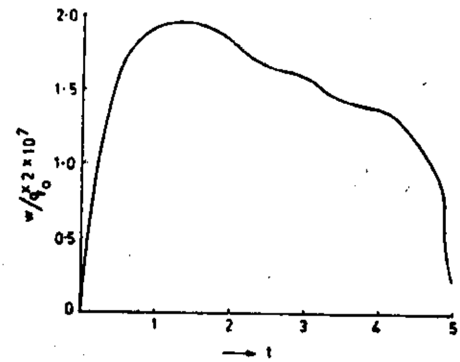

Fig. 5—Centre Deflection vs Time (Clamped)

$$\bar{u}=a/2, \quad \bar{v}=b/2$$
$$\eta=2\text{m}, \quad \epsilon=1.2\text{ m}$$

the other parameters being as before. It is extremely necessary to consider a large number of terms in order to get very accurate results, which is possible only on fast computer. The figures here represent only the qualitative aspect of the dynamic response of the shell.

The solution corresponding to the clamped-boundary-case converges quite fast.

CONCLUSIONS

1. The basic equations of motion of shallow spherical shells rectangular in plan can be reduced to the system (1).
2. The spectre of natural frequencies are to be determined using formula (5), when all the edges are simply-supported, formula (10), when three edges are simply supported and the fourth clamped, and formula (12), when all the edges are clamped.
3. Amplitudes of forced vibration are to be determined using the formula (22) when all edges are simply-supported, and, the formula (25) when all edges are clamped.
4. The dynamic response of thin shallow spherical shells rectangular in plan is completely solved under arbitrary homogeneous boundary conditions.

REFERENCES

1. Bhattacharya, B., "Shallow Spherical Shells" Journal Structural Engineering, Roorkee, Vol. 2, No. 4, Jan. 1974.
2. Bhattacharya, B., Shallow Spherical Shells, rectangular in plan, Journal Structural Engineering, Roorkee, Vol. 2, No. 1, April 1974.
3. Kantorovich, L. V. and Krylov, V. I., "Approximate Methods of Higher Analysis", Interscience, New York, 1964.
4. Marguerre, K., "Zur theorie der gekrummten platte grosser Formanderung", Jahrbuch 1939 der deutschen Luftfahrtforschung.
5. Nowacki, W., "Dynamics of elastic system", Chapman and Hall Ltd., London, 1963.
6. Reissner, E., "On Vibrations of Shallow Spherical Shells", J. Appl. Physics, Vol. 17, No. 12, 1946.
7. Timoshenko, S. P., Vibration Problems in Engineering, 3rd Edition, D. Van Nostrand Co., London.
8. Vlasov, V. Z., "General Theory of Shells and its Applications", NASA, Washington, 1964.