MATHEMATICAL VERIFICATION OF NEWMARK IMPLICIT TIME INTEGRAL FOR APPLICATION IN PSEUDO-DYNAMIC TESTING

K. Sathish Kumar¹, Nagesh R. Iyer¹, K. Muthumani¹ and C. Antony Jeyasehar²

¹CSIR-Structural Engineering Research Centre, Council of Scientific and Industrial Research, Taramani, Chennai 600 113, ksk@serc.res.in, director@serc.res.in, kmm@serc.res.in
²Department of Civil and Structural Engineering, Annamalai University, Annamalai Nagar 608 002, chellamajs@gmail.com

ABSTRACT

In recent years, Pseudo-dynamic (PsD) technique is being adopted as an alternate to conventional shake-table technique to evaluate the seismic performance of structures. The shake-table technique has the merit of simulating all the three force parameters namely inertial, damping and elastic forces in the tested structure realistically; however the technique needs sophisticated shake-table driven by servo controlled actuators with appropriate control electronics. On contrary, PsD technique simulates the three force parameters by using a static actuator through application of an equivalent pseudo-dynamic force system with computation of inertial forces in the back-ground. Such a hybrid technique needs specialized algorithm based on an appropriate mathematical model for the off-line time integration and computation of inertial forces. Several time integrals have been proposed for application in PsD testing and majority of them are derived from Newmark-β family of algorithms. The traditional PsD testing uses constant acceleration version of Newmark time integral in explicit form for mathematical simplicity. This simplified explicit formulation results in numerical damping leading to considerable amplitude error in PsD testing, limiting its application to simple structures. However, for complicated structures improvement is needed in the time integral form leading to unconditional stability and zero numerical damping. This paper presents an improved form of Newmark implicit time integral for PsD testing. The improvement is based on the inclusion of an additional term in displacement predictor, which not only renders the algorithm more consistent, but also eliminates numerical damping and makes the algorithm unconditionally stable. The paper presents the analytical study carried out on the stability and energy dissipation properties of the improved time integral by evaluating its spectral characteristics for verifying its suitability in PsD testing.

Keywords: Pseudo-dynamic testing, shake-table testing, Newmark implicit time integral, numerical damping, numerical stability

INTRODUCTION

Several experimental methods (Kausel, 1998) are used to simulate and evaluate the seismic performance of structures and structural systems. These include, quasi-static testing, effective force testing and shake-table testing. Among them, the shake-table technique has the merit of simulating all
the three force parameters namely inertial, damping and elastic forces in the tested structure realistically; however the technique needs sophisticated shake-table driven by servo controlled actuators with excellent control electronics. In the absence of such an expensive shake-table facility, it is possible to simulate the three force parameters using a static actuator through application of an equivalent pseudo-dynamic force system by computation of inertial forces in the background. For such a hybrid Pseudo-dynamic (PsD) method, a specialized algorithm based on an appropriate mathematical model is needed for the off-line time integration and computation of inertial forces such that the forces are applied statically through static actuators. This alternate seismic performance evaluation methodology is picking up in the recent years and it is essential for to-days needs of growing India with enhanced seismic risk. This method has the advantage of testing large and tall test structures with center of mass well above the base which are normally can not be tested on a shake table for evaluating their seismic performance. As this method involves application of dynamic forces in an equivalent static mean through static actuators, close monitoring of the structural behavior including crack initiation, crack growth and stiffness degradation is also becomes possible. The drawback in such a hybrid method is the lack of simulation of strain rate effects which may not be critical under seismic loads. Also the method is time consuming due to its iterative nature.

PsD Testing Method

PsD testing is a combined computational and experimental technique for evaluating dynamic systems originally proposed by Takanashi et al (1975). The method relies on modeling inertial and damping forces computationally, while the nonlinear restoring forces are measured experimentally. Dynamic equilibrium equations can generally be expressed as

$$M \frac{d^2 x}{dt^2} + C \frac{dx}{dt} + r(x) = f$$  \hspace{1cm} (1)

Where, $M$ and $C$ are mass and viscous damping matrices and $x$, $r$ and $f$ are the displacement, restoring force and applied force vectors respectively. It is assumed that $r(x)$ is the only source of nonlinearity which can be obtained accurate enough through experimental measurements. The PsD test method uniquely utilizes both computational and experimental terms to form the equation of motion (Eq. (1)). The response is obtained by discretising time and calculating it in a step-by-step manner. A time stepping formulation computes a displacement step which is subsequently imposed on the structure by means of computer controlled servo-hydraulic actuators as shown in Fig. 1.

![Diagram of computational and experimental components of a typical pseudo-dynamic (PsD) test set-up](image)

Once the structure has been deformed, the resulting restoring forces are measured. Based on these restoring forces and the current damping and applied forces, the resulting new acceleration may be calculated. A new displacement step can then be calculated, and the next step has thus commenced.
In comparison to shaking table testing, there are some important differences. As the PsD testing is carried out in a step-by-step fashion, it is clear that it is unrealistic to be able to progress the test in real time. Furthermore, as inertial effects are modeled computationally, such forces need not and should not exist in the physical model. The time scale of a typical test is therefore expanded in magnitude which has both beneficial and adverse effects. The fact that the structure is displaced slowly (and can even be stopped) provides a good opportunity for inspection and any detailed readings to be taken; however, the strain rate effects on material response are neglected.

**PsD Formulation based on Newmark Explicit Time Integral**

Several time stepping formulation have been proposed for application in PsD testing (Bursi and Shing, 1996). The majority of them are explicit due to the fact that the nonlinear structural restoring forces at the end of any time step are unknown and displacement iterations in PsD test are undesirable as these might result in partial unload. The integral form of the Newmark explicit was initially proposed by Chang et al (1998) by integrating the equation of motion (Eq. 1) in its incremental form once with respect to time. The solution involves utilization of the time integral of the force for each time step which can be found reasonably accurately through some simple numerical integration and sub-stepping. The basic Newmark implicit relations and integrating Eq. 1 yields:

\[
M\Delta \frac{dx}{dt} + C\Delta x + \Delta \int r(x)\,dt = \Delta \int f\,dt \quad (2a)
\]

\[
d_{n+1} = d_n + \Delta tv_n + \left( \frac{1}{2} - \beta \right)(\Delta t)^2 a_n + \beta(\Delta t)^2 a_{n+1} \quad (2b)
\]

\[
v_{n+1} = v_n + (1 - \gamma)\Delta t a_n + \gamma \Delta ta_{n+1} \quad (2c)
\]

Where, \( \Delta t \) is the time step duration, \( d \) and \( v \) the displacement and velocity, respectively, and \( \Delta \) indicates the change over one time step. The Chang formulation builds on integrating the incremental equations of the explicit format of the Newmark method by using \( \beta = 0 \) in the Newmark implicit relations. Then the basic Newmark explicit equations are

\[
M\Delta a_{n+1} + C\Delta v_{n+1} + \Delta r_{n+1} = \Delta f_{n+1} \quad (3a)
\]

\[
d_{n+1} = d_n + \Delta tv_n + \frac{1}{2}(\Delta t)^2 a_n \quad (3b)
\]

\[
v_{n+1} = v_n + \frac{1}{2} \Delta t(a_n + a_{n+1}) \quad (3c)
\]

Eqs. 3 are integrated once again with respect to time which leads to the following equations:

\[
M\Delta v_{n+1} + C\Delta d_{n+1} + \Delta \int r_{n+1} \,dt = \Delta \int f_{n+1} \,dt \quad (4a)
\]

\[
\int d_{n+1} \,dt = \int d_n \,dt + \Delta td_n + \frac{1}{2}(\Delta t)^2 v_n \quad (4b)
\]

\[
d_{n+1} = d_n + \frac{1}{2} \Delta t(v_n + v_{n+1}) \quad (4c)
\]

In the usual Newmark explicit formulation (Eq. 3), the equations are solved for the change in acceleration, and the equations of motion in the integral form, Eqs. 4, are now solved for the change in velocity. Additionally, the integral form has an expression for the time-integral of displacement instead of the displacement predictor \( \Delta d_{n+1} \) in the usual form. More importantly, the term in the integral form no longer represents an explicit prediction that may be used as an initial displacement step in pseudo-dynamics. The displacement step is now an implicit function of \( v_n \) and \( v_{n+1} \), and can be
found from Eq. 4c, which in turn requires the solution of Eq. 4a to obtain the velocity at the end of the time step \( v_{n+1} \). In effect, the action of integrating the set of equations has rendered the method implicit in the sense that the predictor displacement cannot be deduced directly any more. The integral form formulation also requires an assessment of the integral of the restoring force before the displacement predictor can be calculated. Such an estimate enables the solution for \( \Delta v_{n+1} \), to be found, which in turn produces an explicit predictor for the displacement Eq. 4c which is needed for the PsD implementation. The restoring force and its time integral are nonlinear functions of displacement, and can no longer be obtained directly, as no predictor displacement step exists to be imposed. In order to be able to utilize the formulation, Chang et al (1998) suggest multiplying Equation 4b by the tangent stiffness, and an explicit expression of the integral of the restoring force at \( t = t_{n+1} \) may be found (here expressed for an SDOF system) as outlined in Eq. 5.

\[
\int_{r_{n+1}} dt = \int r_t dt + \Delta tkd_n + \frac{k}{2} (\Delta t)^2 v_n = \int r_t dt + \Delta tr_n + \frac{k}{2} (\Delta t)^2 v_n \quad (5)
\]

The physical interpretation of the Eq. 5 indicates the restoring force time area at a given time \( t \), while the sum of the two remaining terms represents the projected trapezoidal area assuming that a constant velocity exists until the end of the step. Such a procedure tentatively assumes that the tangent stiffness is known, or may be obtained somehow, which will normally not be the case in PsD testing. To overcome the problem of the unknown tangent stiffness matrix, Chang et al (1998) suggest replacing it with the initial stiffness term. The error involved is not large as the tangent stiffness is required only in the second order term on the right hand side of Eq. 5. In any case, once an expression for the integral of the restoring force exists, Chang et al (1998) suggest a solution procedure where they solve for \( \Delta v_{n+1} \) from Eq. 4a by substituting Eq. 4c for \( d_{n+1} \). When considering an SDOF system, the velocity change can be expressed as

\[
\Delta v_{n+1} = \left( m + \frac{\Delta t}{2} c \right)^{-1} \left( \Delta \int f_{n+1} dt - c \Delta v_n - \Delta tr_n - \frac{k}{2} (\Delta t)^2 v_n \right) \quad (6)
\]

Where, \( k_0 \) is the initial stiffness in place of the tangent stiffness term. By substituting this result back into Eq. 4c, a prediction for the change in displacement can finally be expressed as follows:

\[
\Delta d_{n+1} = \Delta v_n + \frac{\Delta t}{2} \left( m + \frac{\Delta t}{2} c \right)^{-1} \left( \Delta \int f_{n+1} dt - c \Delta v_n - \Delta tr_n - \frac{k}{2} (\Delta t)^2 v_n \right) \quad (7)
\]

Which, when added on to the previous displacement value, furnishes an explicit displacement predictor to be applied in pseudo-dynamic tests similarly to any other displacement predictor. In general, due to the material nonlinearity, the restoring force will not follow the linear extrapolation as estimated, and the change in the time integral of this restoring force will in reality be smaller than estimated. For this reason, Chang’s algorithm then recalculates \( \Delta v_{n+1} \) based on the measured time integral of the restoring force.

**PsD Formulation based on Newmark Implicit Time Integral**

There is an inconsistency in the integral form of the Newmark explicit formulation when recalculating \( \Delta v_{n+1} \) based on the updated \( \Delta r_{n+1} dt \). \( \Delta v_{n+1} \) indeed has to be recalculated; otherwise, the information about the experimentally measured restoring forces is never taken into account. The predictor \( \Delta d_{n+1} \) is based on an estimation of \( \Delta v_{n+1} \) which in turn build on an approximation of the restoring force as expressed in Eq. 5. Once the integral of the restoring force has been obtained, the integrated equation of motion, Eq. 4a, can be applied. Expanding Equation 4c yields:
\[ d_{n+1} = d_n + \frac{1}{2} \Delta t (v_n + v_{n+1}) = d_n + \frac{1}{2} \Delta t (v_n + \Delta v_{n+1}) = d_n + \Delta t v_n + \frac{1}{2} \Delta t \Delta v_{n+1} \] (8)

Furthermore, Eq. 4a assumes the integral of the restoring force over that time step, \( \Delta \int r_{n+1}/dt \), to be determined by computing the time integral of the restoring force over the time step. Assuming linear stiffness for simplicity, the exact expression for \( \Delta \int r_{n+1}/dt \) will be as follows:

\[ \Delta \int r_{n+1}/dt = \int r_{n+1}/dt - \int r_n/\ dt = \Delta tk \left( \frac{d_n + d_{n+1}}{2} \right) \] (9)

Eq. 8 can now be substituted for \( d_{n+1} \) in Equation 9 to yield the following:

\[ \Delta \int r_{n+1}/dt = \Delta tk \left( \frac{d_n + d_n + \Delta t v_n + 1/2 \Delta t \Delta v_{n+1}}{2} \right) \] (10)

This can be further manipulated into

\[ \Delta \int r_{n+1}/dt = \Delta t k d_n + \frac{1}{2} \Delta t^2 k v_n + \frac{1}{4} \Delta t^2 k \Delta v_{n+1} \] (11)

By comparing Eq. 11 with Eq. 5, it is clear that Eq. 11 now contains one additional term, \( (\Delta t^2 k \Delta v_{n+1})/4 \). This term corresponds to the term which is omitted in the standard implicit Newmark algorithm to render it explicit (i.e., \( \beta = 0 \)), and the omission of this term is the cause of the numerical damping invariably present in the integral form of the method. However, omitting the equivalent term in the integral form of the method does not render the method explicit; in fact, its omission has no bearing on the nature of the algorithm. The integral form of the Newmark algorithm has been made explicit through an estimation of the time integral of the restoring force, which enables the calculation of \( \Delta v_{n+1} \) followed by \( \Delta d_{n+1} \). At this point, it becomes clear that there is no reason why the seemingly implicit additional term in Eq. 11 cannot be included in the estimation of the time integral of restoring force, as the implicit variable is the actual unknown, the expression is trying to represent. The situation is clarified through the following argument, where Eq. 11 has been substituted into Eq. 4a and solved for \( \Delta v_{n+1} \) to yield an alternative expression for Eq. 6.

\[ \Delta v_{n+1} = \left( m + \frac{\Delta t}{2} c \right)^{-1} \left( \Delta \int f_{n+1}/dt - c \Delta v_n - \Delta t r_n - \frac{k_0}{2} (\Delta t)^2 v_n - \frac{1}{4} \Delta t^2 k_0 \Delta v_{n+1} \right) \] (12)

The unknown, \( \Delta v_{n+1} \), is present on both sides of the equation, but through further rearrangement

\[ \Delta v_{n+1} = \left( m + \frac{\Delta t}{2} c \right)^{-1} \frac{1}{4} \Delta t^2 k_0 \Delta v_{n+1} \]

\[ = \left( m + \frac{\Delta t}{2} c \right)^{-1} \left( \Delta \int f_{n+1}/dt - c \Delta v_n - \Delta t r_n - \frac{k_0}{2} (\Delta t)^2 v_n \right) \left( 1 + \left( m + \frac{\Delta t}{2} c \right)^{-1} \frac{1}{4} \Delta t^2 k_0 \right) \Delta v_{n+1} \]

\[ = \left( m + \frac{\Delta t}{2} c \right)^{-1} \left( \Delta \int f_{n+1}/dt - c \Delta v_n - \Delta t r_n - \frac{k_0}{2} (\Delta t)^2 v_n \right) \]

an explicit expression for the velocity can finally be found.
\[ \Delta v_{n+1} = \left( m + \frac{\Delta t}{2} c \right)^{-1} \left( \Delta t \int f_{n+1} dt - c \Delta t v_n - \Delta t r_n - \frac{k_0}{2} (\Delta t)^2 v_n \right) \]

\[ 1 + \left( m + \frac{\Delta t}{2} c \right)^{-1} \frac{1}{4} \Delta t^2 k_o \] (13)

This equation may now be substituted directly into Eq. 8, and rearranged to obtain a new explicit expression for the displacement predictor \( \Delta d_{n+1} \):

\[ \Delta d_{n+1} = \Delta v_n + \frac{\Delta t}{2} \left( m + \frac{\Delta t}{2} c \right)^{-1} \left( \Delta t \int f_{n+1} dt - c \Delta t v_n - \Delta t r_n - \frac{k_0}{2} (\Delta t)^2 v_n \right) \]

\[ 1 + \left( m + \frac{\Delta t}{2} c \right)^{-1} \frac{1}{4} \Delta t^2 k_o \] (14)

By using Eq. 14 rather than Eq. 7 as the displacement predictor, the time stepping formulation has become implicit. Principal differences between the two PsD formulations are summarized in Table 1.

**Table 1 Comparison of Chang’s PsD formulation based on Newmark explicit time integral and proposed improved PsD formulation based on Newmark implicit time integral**

<table>
<thead>
<tr>
<th>Predictor ( d_{n+1} )</th>
<th>Chang’s PsD formulation based on Newmark explicit time integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int d_{n+1} dt )</td>
<td>( \int d_n dt + \Delta t d_n + \frac{1}{2} (\Delta t)^2 v_n )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Predictor ( d_{n+1} )</th>
<th>Proposed improved PsD formulation based on Newmark implicit time integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int d_{n+1} dt )</td>
<td>( \int d_n dt + \Delta t d_n + \frac{1}{2} (\Delta t)^2 v_n + \frac{1}{4} (\Delta t)^2 \Delta v_{n+1} )</td>
</tr>
</tbody>
</table>

**MATHEMATICAL VERIFICATION OF THE IMPROVED PSD FORMULATION**

The suitability of the proposed PsD formulation based on Newmark implicit relations is verified through evaluating its stability and numerical damping properties. The stability properties of a time integral formulation are studied by considering the spectral properties of its recursive amplification matrix. Considering the state vector of the system at time \( t = t_n \), the integral form of the numerical time integral form yields:

\[ X_n = \left[ \begin{array}{c} \int d_n dt \\ \Delta t d_n \\ \Delta t^2 v_n \end{array} \right] \] (15)
For stability purposes, one can ignore the external load vector and damping forces, thus the recurrent relationship between the state vector at \( t = t_n \) and \( t = t_{n+1} \) can be expressed in terms of recursive amplification matrix \([A]\) as

\[
X_{n+1} = [A]X_n \quad (16)
\]

**Stability and Dissipation Properties of PsD Formulation based on Newmark Explicit Time Integral**

Considering first the explicit time integral form, Eqs. 4 can be expressed in terms of the variables of the state vector as

\[
\int d_{n+1} \, dt = \int d_n \, dt + \Delta t \, d_n + \frac{1}{2} (\Delta t)^2 v_n
\]

\[
d_{n+1} = d_n + \frac{1}{2} \Delta t (v_n + v_{n+1}) \quad (17)
\]

\[
\Delta v_{n+1} = M^{-1} \left[ n+1 - \int r_{n+1} \, dt \right]
\]

Depending on the precise implementation of the time integral form, i.e., whether \( d_{n+1} \) is updated following the recalculation of \( v_{n+1} \) or not, the exact expressions for \( d_{n+1} \) and \( r_{n+1} \, dt \) will differ. Assuming initially that \( d_{n+1} \) is not recalculated, the predicted \( d_{n+1} \) remains, and the term \( r_{n+1} \, dt \) will be a function of the restoring force both at the start and at the end of the predicted step yielding the second of Eqs. 18. \( d_{n+1} \) will thus no longer be represented by the implicit expression in Eq. 17, rather by a simplification of Eq. 7 containing only the terms relevant for stability analyses. \( r_{n+1} \, dt \) may be defined in terms of \( d_{n+1} \) or by the expression for the prediction step; however, this will in the end lead to the same amplification matrix. Assuming linear stiffness for simplicity, \( r_{n+1} \, dt \) may be expressed as \( \frac{\Delta t k}{2} (d_n + d_{n+1}) \), yielding the third equation of Eqs. 18.

\[
\int r_{n+1} \, dt = \int r_n \, dt + \Delta t d_n + \frac{1}{2} (\Delta t)^2 v_n
\]

\[
d_{n+1} = d_n + \Delta t v_n + \frac{1}{2} \Delta t M^{-1} \left( -\Delta t k d_n - \frac{k}{2} (\Delta t)^2 v_n \right) \quad (18)
\]

\[
v_{n+1} = v_n + M^{-1} \left( -\frac{\Delta t k}{2} (d_n + d_{n+1}) \right)
\]

Multiplying the second and third equations by \( \Delta t \) and \( \Delta t^2 \) respectively and substituting \( \Omega^2 \) and \( \Delta t^2 k/m \) yield Eqs. 19.

\[
\int r_{n+1} \, dt = \int r_n \, dt + \Delta t d_n + \frac{1}{2} (\Delta t)^2 v_n
\]

\[
\Delta t d_{n+1} = \Delta t d_n + \Delta t^2 v_n + \frac{1}{2} \Omega^2 v_n - \frac{1}{4} \Omega^2 (\Delta t)^2 v_n \quad (19)
\]

\[
\Delta t^2 v_{n+1} = \Delta t^2 v_n - \Omega^2 \frac{\Delta t d_n}{2} - \Omega^2 \frac{\Delta t d_{n+1}}{2}
\]

After sorting terms at \( t = t_{n+1} \) and \( t = t_n \) and expressing them in matrix form the amplification matrix is obtained as
This clearly differs from the normal Newmark explicit matrix. Stability of a time integral formulation is ensured when the spectral radius of the amplification matrix does not exceed unity. In the above matrix, one eigen value will be equal to unity, while the other two will form a pair of complex conjugates. Corresponding moduli have been plotted below as a function of $\Omega$ in Fig. 2.

\[
A = \begin{bmatrix}
1 & 1 & \frac{1}{2}
\frac{1}{4} & \frac{1}{8}
0 & 1 - \frac{\Omega^2}{2} & 1 - \frac{\Omega^2}{4}
0 & -\Omega^2 + \frac{\Omega^2}{4} & 1 - \frac{\Omega^2}{2} + \frac{\Omega^2}{8}
\end{bmatrix} \tag{20}
\]

From Fig. 2, both the expected stability limit of 2.0 and the existence of noticeable numerical damping can be seen for the values of $\Omega$ exceeding 0.5. If however the $d_{n+1}$ is recalculated once the corrected $v_{n+1}$ has been found, the situation is somewhat different. Eqs. 18 will be altered such that the displacement predictor is no longer present in the definition of $d_{n+1}$, but remains in the expression for $v_{n+1}$. This yields Eqs. 21, which can be represented by the amplification matrix shown in Eq. 22, obtained the same way as Eq. 20. This time integral form exhibits similar stability and damping characteristics as the standard Newmark explicit; perfect energy conservation up to the stability limit of 2.0. However, as the algorithm stands, it cannot be directly implemented into a PsD test. This is because $r_n$ is in fact unknown at the start of the time step. The reason for this is that $d_n$ was recalculated after the completion of the previous step, and the restoring force caused by it is thus unknown.

\[
\int r_{n+1} dt = \int r_n dt + \Delta t d_n + \frac{1}{2} (\Delta t)^2 v_n
\]

\[
d_{n+1} = d_n + \frac{1}{2} \Delta t v_n + \frac{1}{2} \Delta t v_{n+1} \tag{21}
\]

\[
v_{n+1} = v_n + M^{-1} \left( \frac{\Delta t k}{2} \left( 2d_n + \Delta t v_n + \frac{1}{2} \Delta t M^{-1} \left( -\Delta t k d_n - \frac{k}{2} (\Delta t)^2 v_n \right) \right) \right)
\]
\[
A = \begin{bmatrix}
1 & 1 & 1 \\
0 & -\frac{\Omega^2}{2} + \frac{\Omega^4}{8} & -\frac{\Omega^2}{4} + \frac{\Omega^4}{16} \\
0 & -\frac{\Omega^2}{2} + \frac{\Omega^4}{4} & -\frac{\Omega^2}{2} + \frac{\Omega^4}{8}
\end{bmatrix}
\] (22)

The correct procedure would require the recalculated \(d_{n+1}\) to be imposed separately, and the corresponding restoring force re-measured. Such a procedure would lead to a double step implementation, but employing iterations in a PsD algorithm that is still only conditionally stable seems inappropriate.

**Stability and Dissipation properties of the Improved PsD Formulation based on Newmark Implicit Time Integral**

The effects of using the implicit version of the time integral form with the modified displacement predictor are substantial. Not only does the method avoid the numerical damping associated with the Newmark explicit – integral form algorithm, but owing to the fact that the algorithm is now genuinely implicit, it also becomes unconditionally stable. By considering the expression for the time integral of displacement, the displacement and the velocity and using a similar procedure as with Eq. 19, it leads to the following equations:

\[
[\dot{d}_{n+1}] = [\dot{d}_n] + \Delta [d_n] + \frac{1}{2}h[\Delta t]^2v_n - \beta \frac{\Omega^2}{1 + \Omega^2/4} \Delta [d_n] - \frac{1}{2} \beta \frac{\Omega^2}{1 + \Omega^2/4} (\Delta t)^2 v_n
\]

\[
\Delta [d_{n+1}] = \Delta [d_n] + \Delta t \gamma v_n - \gamma \frac{\Omega^2}{1 + \Omega^2/4} \Delta [d_n] - \frac{1}{2} \gamma \frac{\Omega^2}{1 + \Omega^2/4} (\Delta t)^2 v_n
\]

\[
\Omega^2 [\dot{d}_{n+1}] + (\Delta t)^2 [d_{n+1}] = \Omega^2 [\dot{d}_n] + (\Delta t)^2 [d_n]
\]

Where, \(\beta\) and \(\gamma\) are the parameters normally present in the Newmark algorithms which typically take the values of 0.25 and 0.50 respectively. Expressing the above equations in a matrix form yields again the recursive amplification matrix of the integration operator:

\[
A = \begin{bmatrix}
1 & 1 - \beta & 1 - \frac{1}{2} \beta \\
0 & 1 - \gamma & 1 - \frac{1}{2} \gamma \\
0 & -\Omega^2 (1 - \beta) & 1 - \Omega^2 (1 - \beta)
\end{bmatrix}
\] (24)

The complex expression for the eigen values of \([A]\) can be simplified to

\[
(1 - \lambda) [\lambda^2 + \left(-2 + \frac{\Omega^2}{2}\lambda\right) \left(1 - \frac{1}{4} \xi^2\right)] \lambda + 1 + \frac{\Omega^2}{2} - \frac{\Omega^2 \xi^2}{8} - \frac{1}{2} \xi^2 = 0
\]

(25)

Where, \(\lambda\) are the eigen values and \(\xi^2\) is expressed as \(\xi^2 = \frac{\Omega^2}{1 + \frac{1}{4} \Omega^2}\) (26)

Eliminating \(\lambda_1 = 1\) leaves the remaining second order equation:
\[ \lambda_{2,3} = \pm \frac{1}{2} \sqrt{-4 \Omega^2 + \frac{1}{4} \Omega^4 + \frac{1}{4} \Omega^2 \xi_2^2 + \frac{\Omega^4 \xi_2^4}{8} - \frac{\Omega^6 \xi_2^4}{8} + \frac{\Omega^8 \xi_2^4}{64}} \]  

By plotting the real and imaginary parts of the solution and computing the moduli, it can be seen that the moduli for \( \lambda_{2,3} \) are also equal to unity for all \( \Omega \), as shown in Fig. 3. This clearly yields the overall solution that \( \rho(A) = 1 \) for all \( \Delta t \), which implies unconditional stability and perfect energy conservation without any numerical damping. The modifications carried out on the Newmark explicit time integral form have shown to eliminate the numerical damping (or) amplitude error of the algorithm, and also improve the stability properties such that it is now unconditionally stable (Satish Kumar et al, 2009). Hence, the proposed algorithm based on Newmark implicit time integral form found to be more appropriate for implementation in PsD testing applications.

CONCLUSIONS

The genesis, development and mathematical formulation of pseudo-dynamic (PsD) testing for experimental seismic performance evaluation of structures are presented in the paper in detail. The suitability of Newmark implicit time integral for PsD testing application towards seismic performance evaluation of structures is verified numerically through studying its stability and numerical damping characteristics. The study showed that Newmark implicit time integral is found to have unconditional stability and zero numerical damping resulting in near-zero amplitude error in PsD testing.

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REFERENCES